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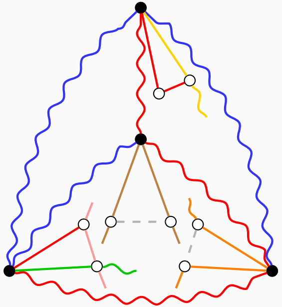
Gallai's Path Decomposition in Planar Graphs

PhD defense of Alexandre Blanché

December 13, 2021



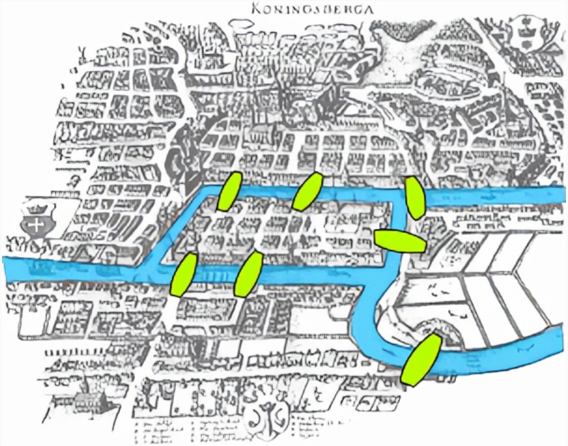
Advisors:
Marthe Bonamy, Nicolas Bonichon

Under examination of:
Stéphane Bessy . . . . Reviewer
Fábio Botler . . . . Invited Member
Nadia Brauner . . . . Examinator
Paul Dorbec . . . . Reviewer
Arnaud Pêcher . . . . Examinator

Context of Gallai’s conjecture

(1736-1968)

Euler (1736): *Königsberg Bridge Problem*



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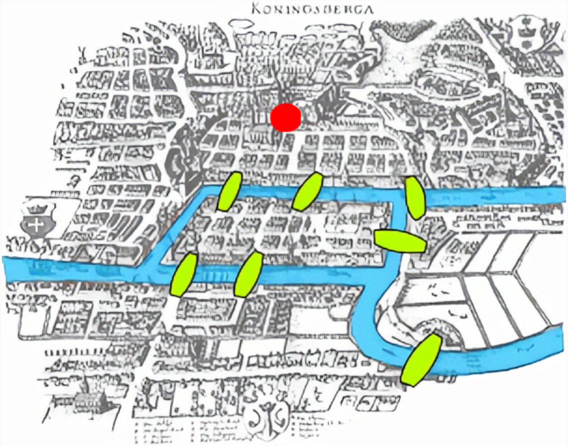

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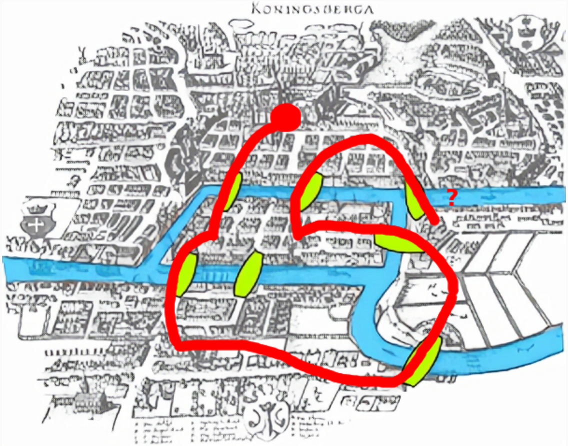

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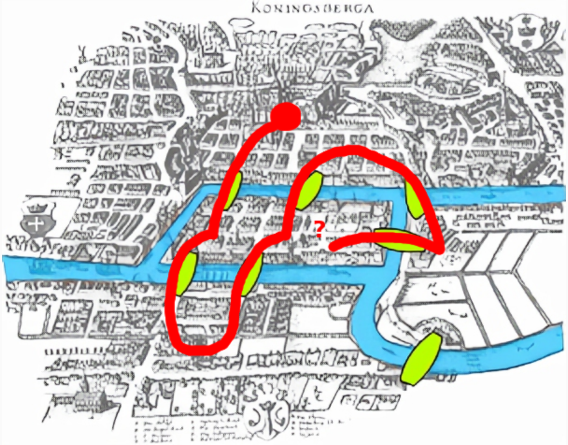

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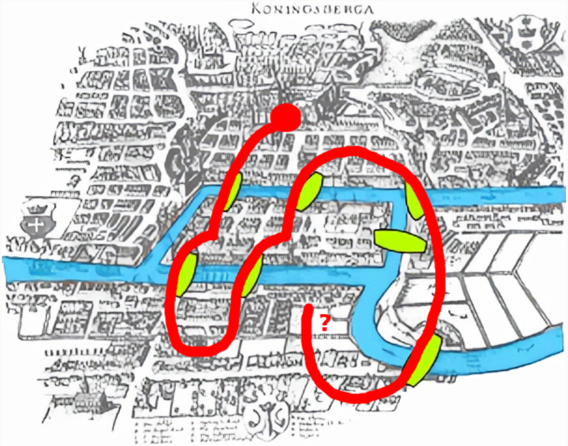

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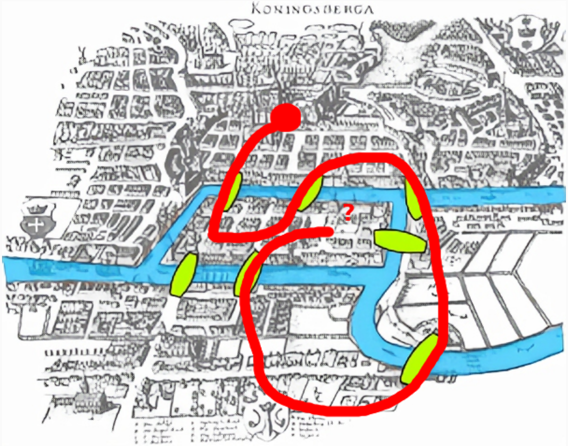

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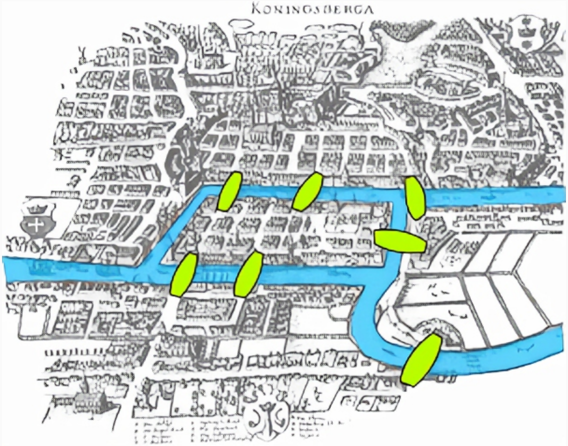

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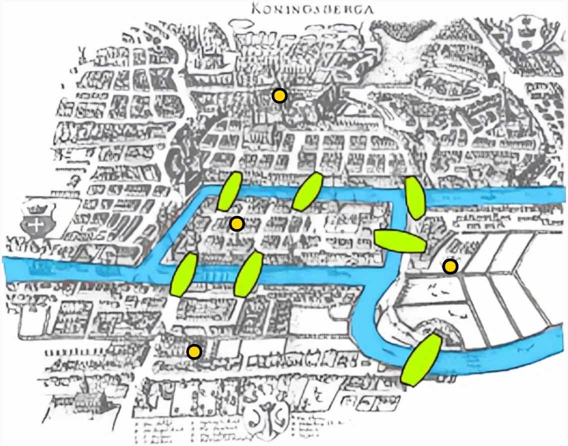

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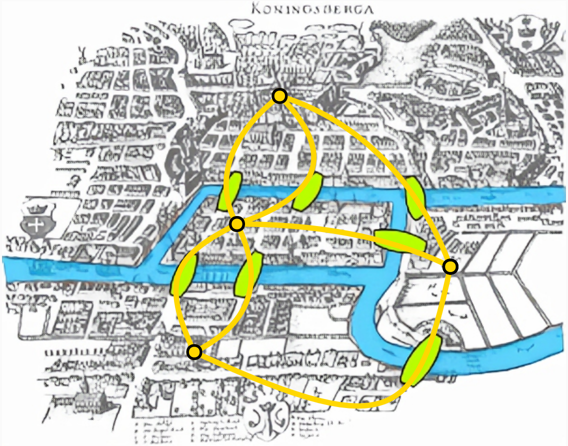

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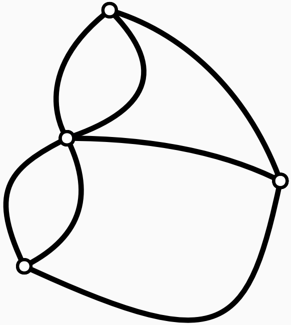

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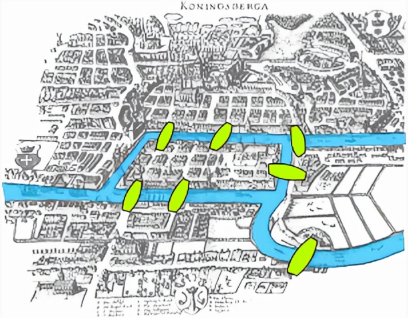

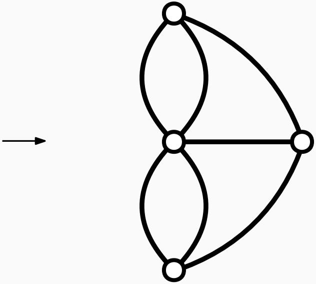

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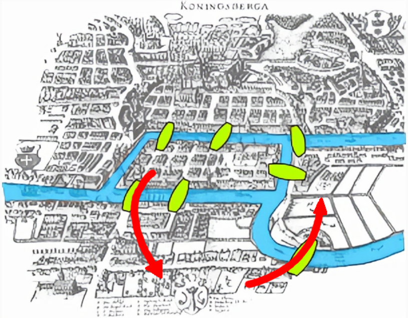

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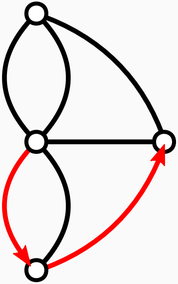
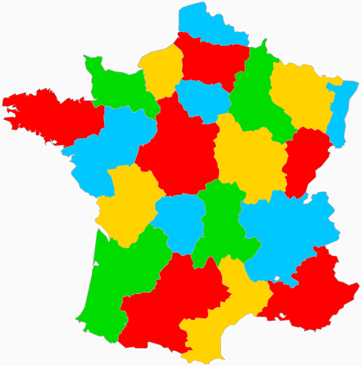


Four-Color Problem

Guthrie, De Morgan (1852)

Can we color the regions of a map with 4 colors, such that two regions that share a border have a different color?



Proved in 1977, as the **Four-color theorem** **[Appel, Haken, 1977]**

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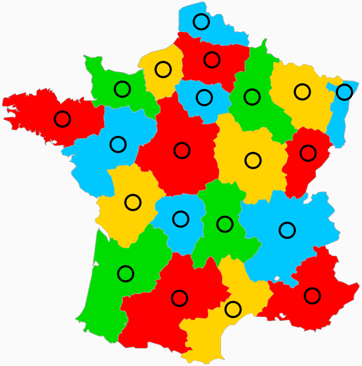

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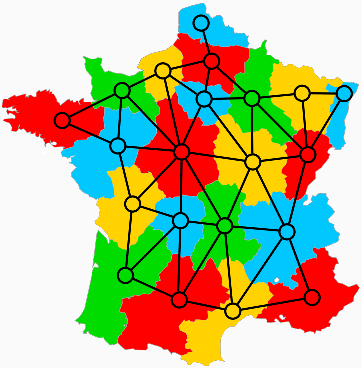
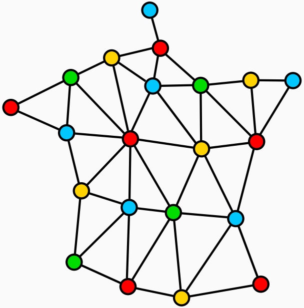


Proved in 1977, as the **Four-color theorem** **[Appel, Haken, 1977]**

Four-Color Problem

*Can we color the **vertices** of a **planar graph** with 4 colors, such that adjacent vertices receive different colors?*



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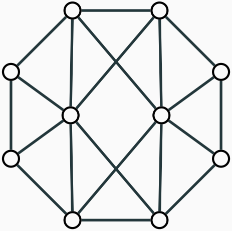
Path decomposition

Path decomposition: a **partition** of the edges into paths



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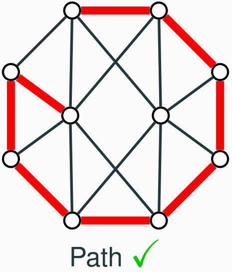

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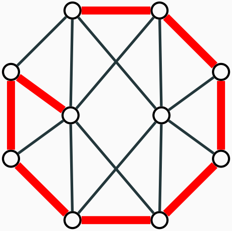
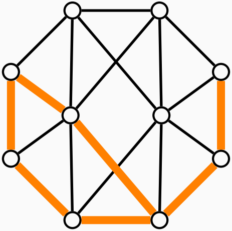


Path ✓



Not a path X

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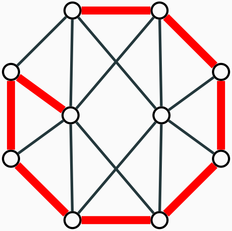

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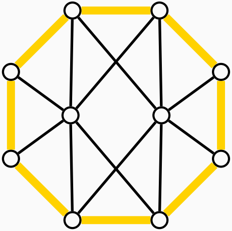

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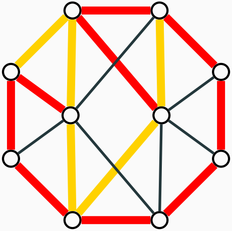

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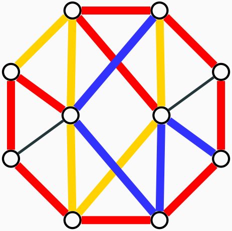

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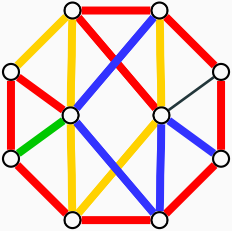

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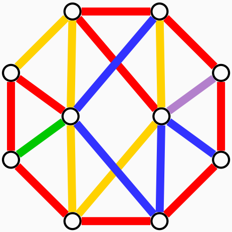

5 colors

Path decomposition

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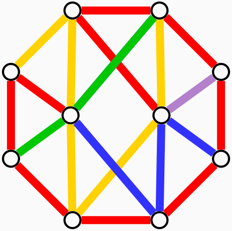

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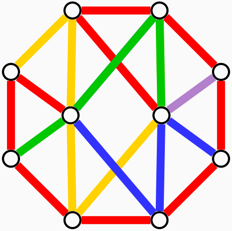

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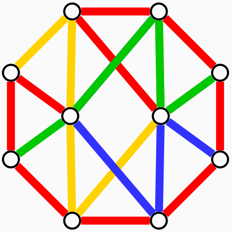

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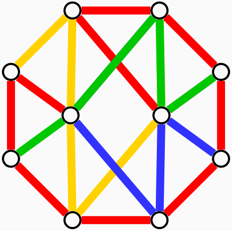

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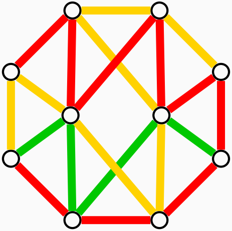
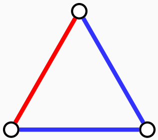


3 colors

Path decomposition



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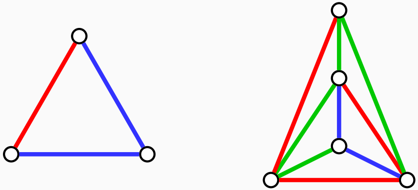

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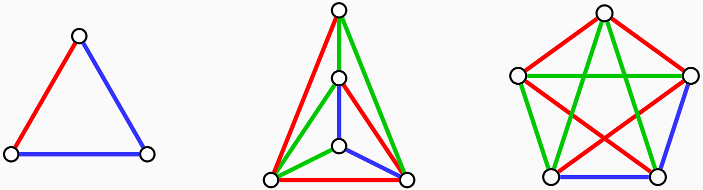

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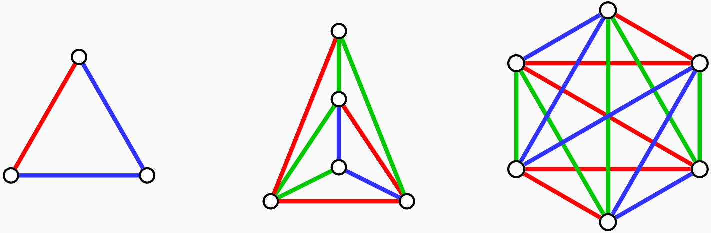

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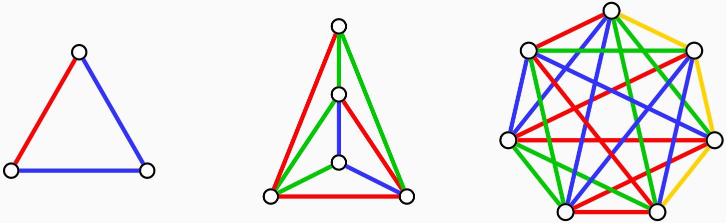

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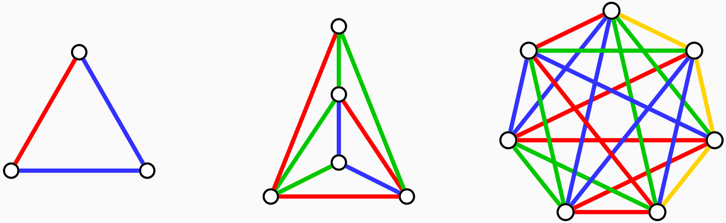
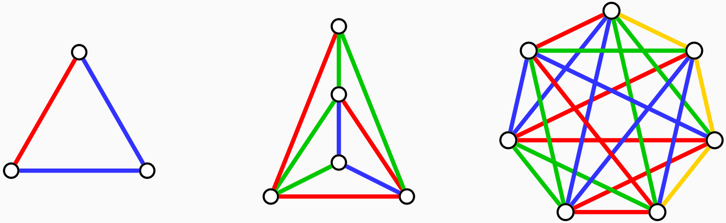


Conjecture (Gallai, 1968)

An n-vertex connected graph has a decomposition into $\leq \left\lceil \frac{n}{2} \right\rceil$ **paths**.

Path decomposition



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Theorem [B., Bonamy, Bonichon, 2021+]

Gallai's conjecture is true on **planar** graphs.

Related conjectures

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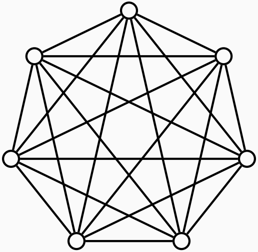
An n-vertex **connected** graph has a decomposition into $\leq \left\lceil \frac{n}{2} \right\rceil$ **paths**.

Conjecture (Hajós, 1968)

An n-vertex **even** graph has a decomposition into $\leq \left\lfloor \frac{n}{2} \right\rfloor$ **cycles**.



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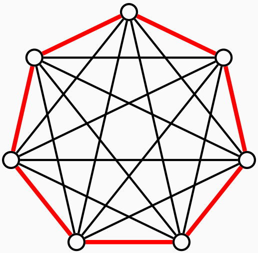

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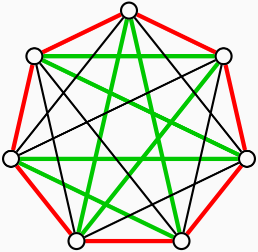

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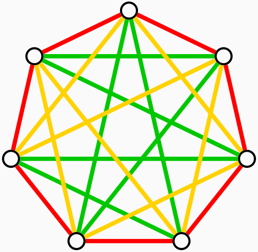
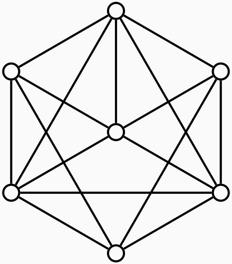


Lovász's initial result

Theorem [Lovász, 1968]

An n-vertex graph has a decomposition into $\leq \left\lfloor \frac{n}{2} \right\rfloor$ **paths and cycles**.



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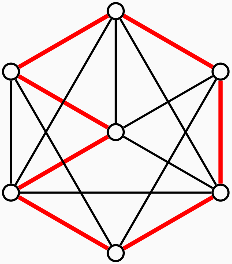

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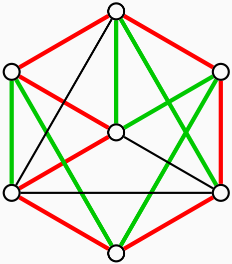

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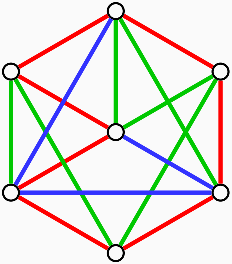


Partial results on Gallai’s conjecture

(1968-2021)

General bounds

Theorems

Any connected graph G has a decomposition into at most $\mathcal{P}(G)$ paths.

|odd|, |even| : number of vertices of odd, even degree of G

- **[Lovász, 1968]**: $\mathcal{P}(G) \leq \frac{|\text{odd}|}{2} + |\text{even}| - 1$
- **[Donald, 1980]**: $\mathcal{P}(G) \leq \frac{|\text{odd}|}{2} + \left\lfloor \frac{3}{4} |\text{even}| \right\rfloor$
- **[Yan, 1998]**, **[Dean, Kouider, 2000]**:

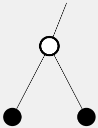
$$\mathcal{P}(G) \leq \frac{|\text{odd}|}{2} + \left\lfloor \frac{2}{3} |\text{even}| \right\rfloor$$

Example: Gallai's conjecture holds on trees

***Reducibility* lemma**

A **minimum counterexample** to Gallai's conjecture on trees **does not contain** a configuration:

- A: 2 leaves with a common parent



A

Example: Gallai's conjecture holds on trees

***Reducibility* lemma**

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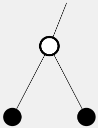
- A: 2 leaves with a common parent
- B: 1 leaf with a parent of degree 2



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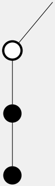

B

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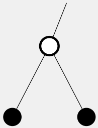

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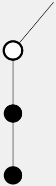

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***Unavoidability* lemma**

All trees with $n \geq 3$ vertices **contain** a configuration A or B.

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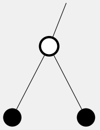

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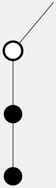

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Unavoidability **lemma**

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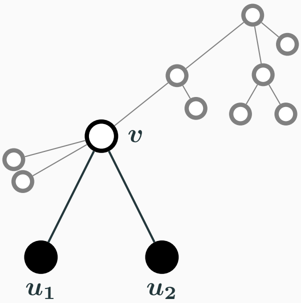
Contradiction $\Rightarrow$ there is no counterexample

Example: Gallai's conjecture holds on trees

In a **minimum counterexample** to Gallai's conjecture on trees:

- Configuration A is impossible:



Configuration *A*

Example: Gallai's conjecture holds on trees

In a **minimum counterexample** to Gallai's conjecture on trees:

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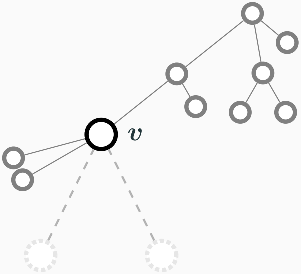

Configuration A

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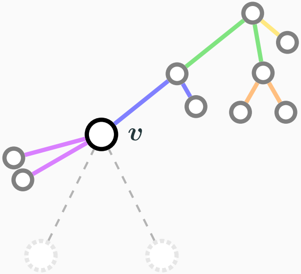

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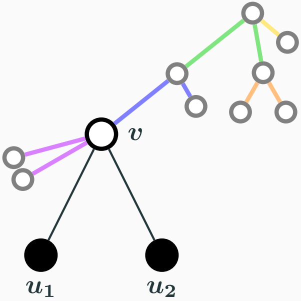

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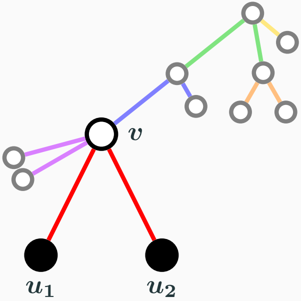

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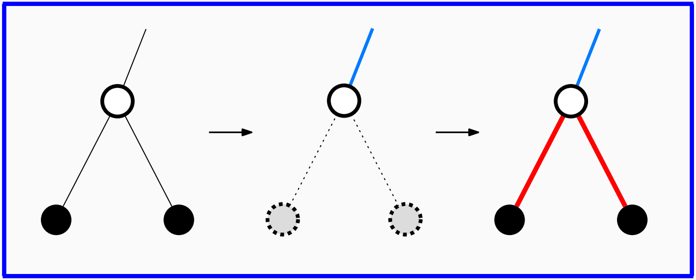
R_A



Configuration A

Example: Gallai's conjecture holds on trees

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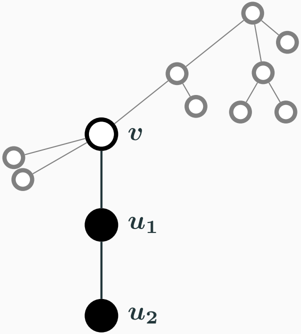

Configuration B

Example: Gallai's conjecture holds on trees

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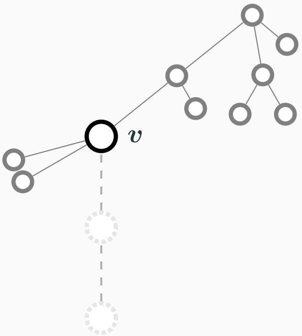

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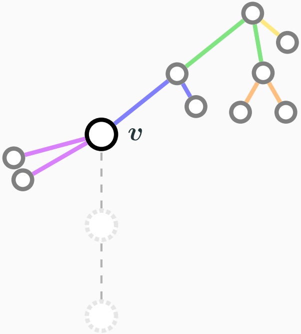

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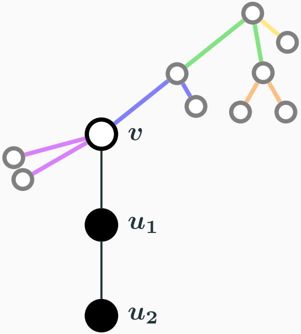

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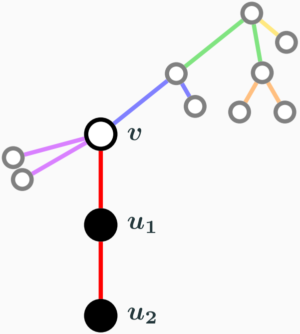

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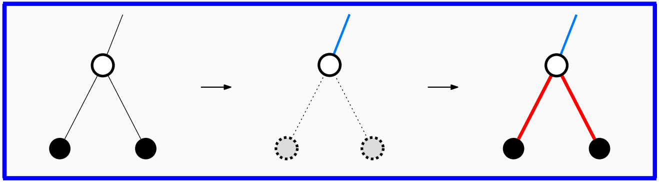
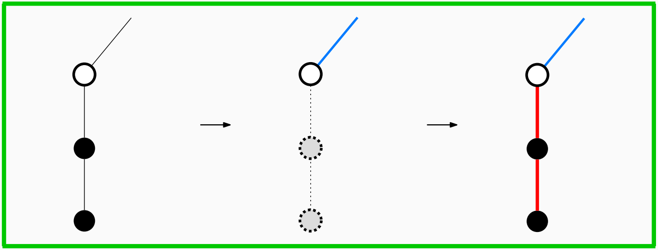


Configuration A

R_B



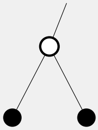
Configuration B

Example: Gallai's conjecture holds on trees

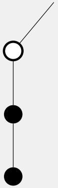
***Reducibility* lemma**

A **minimum counterexample** to Gallai's conjecture on trees **does not contain** a configuration:

- A: 2 leaves with a common parent
- B: 1 leaf with a parent of degree 2



A



B

***Unavoidability* lemma**

All trees with $n \geq 3$ vertices **contain** a configuration A or B.

Contradiction $\Rightarrow$ there is no counterexample

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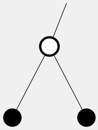

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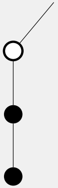

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***Unavoidability* lemma easy** ✓

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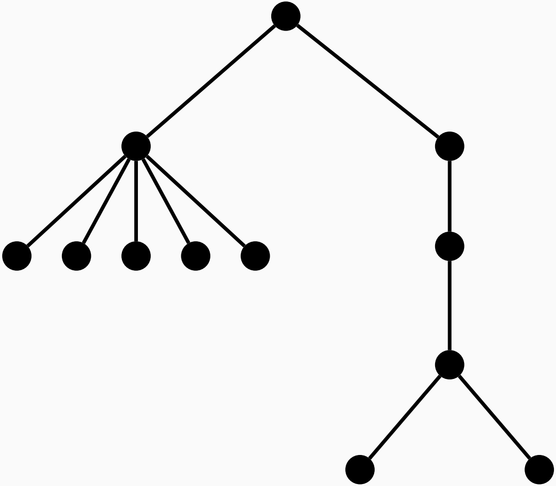
Algorithm for the trees



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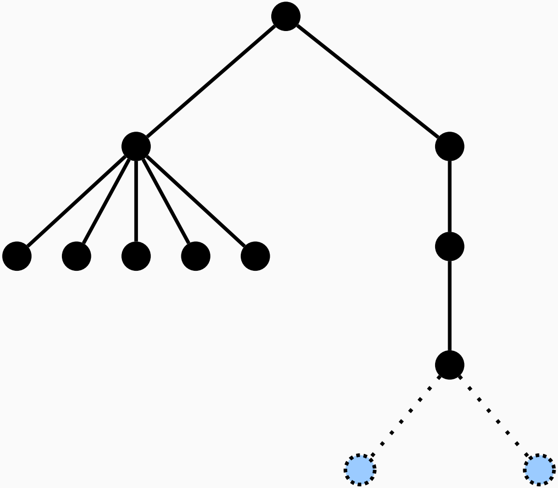

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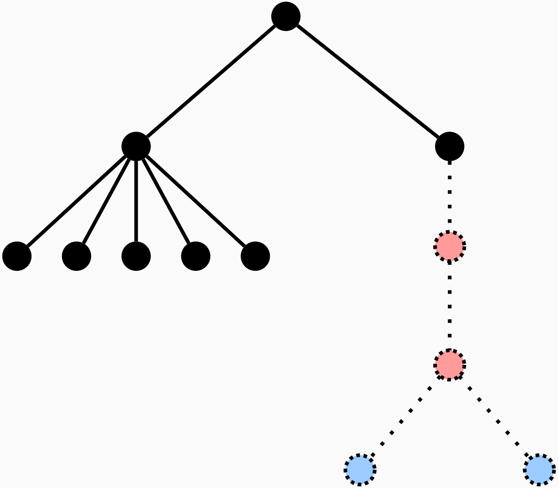

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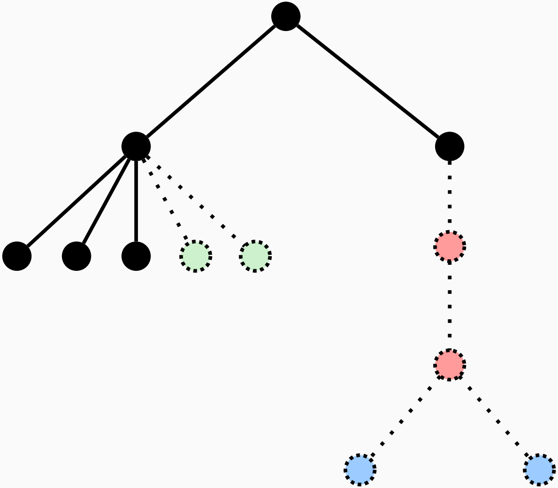

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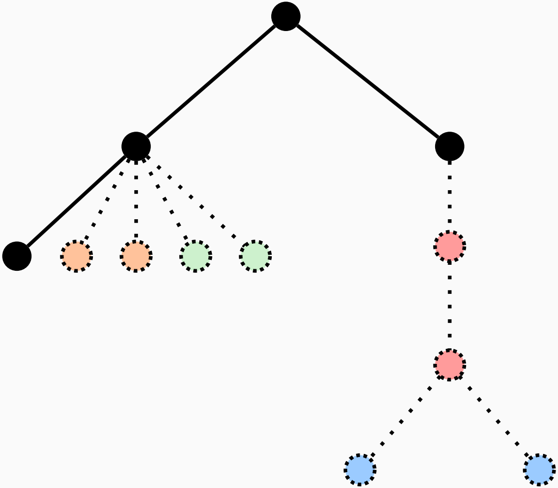

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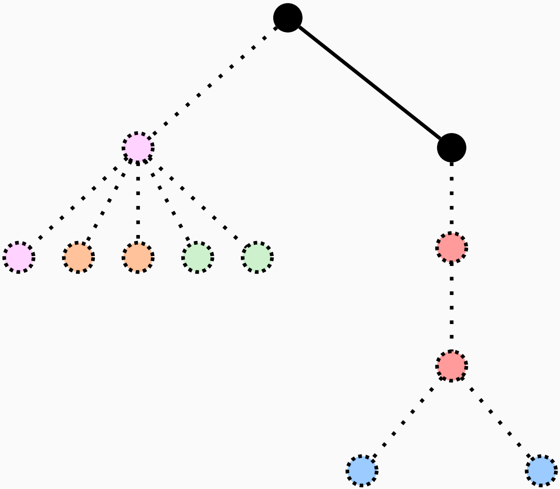

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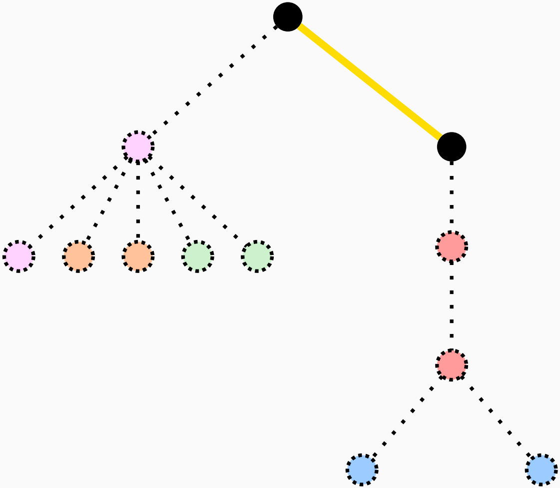

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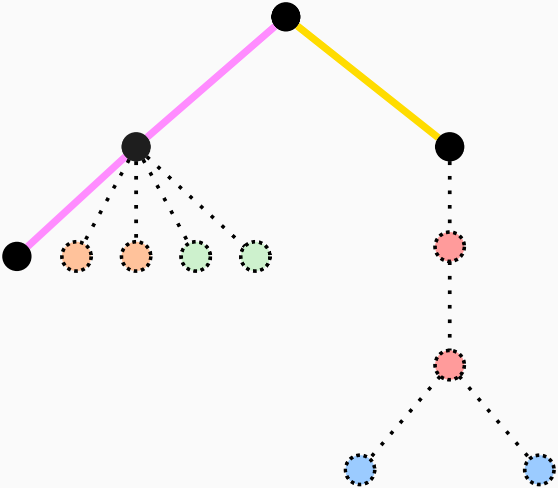

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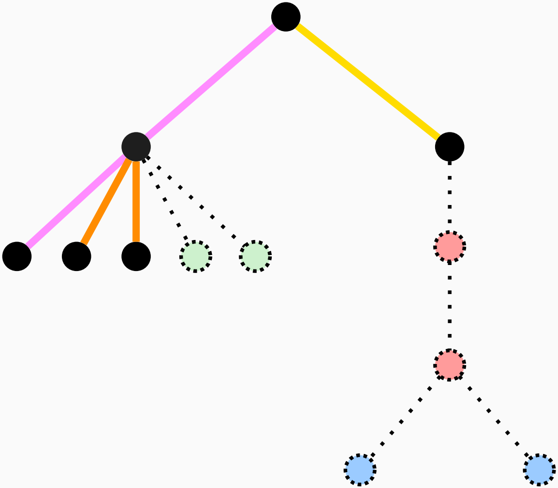

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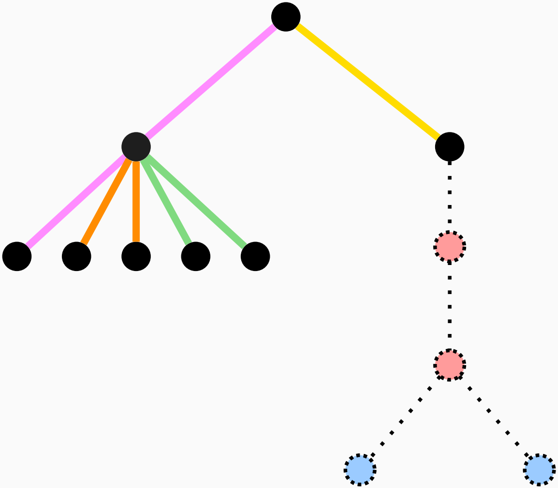

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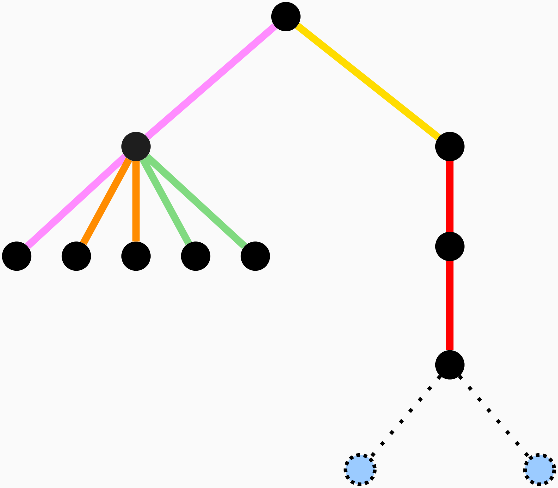

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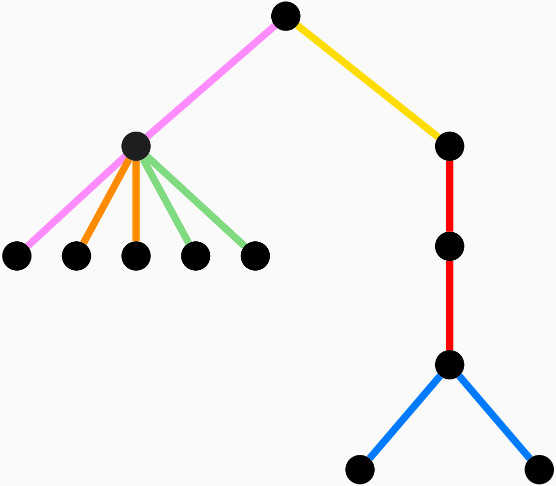

Graph classes on which Gallai's conjecture holds

Even subgraph **(G_{even} = graph induced by vertices of even degree)**

- **[Lovász, 1968]**: $|G_{\text{even}}| \leq 1$
- **[Favaron, Kouider, 1988]**: Each vertex has degree 2 or 4
- **[Pyber, 1996]**: G_{even} is a forest
- **[Fan, 2005]**: Each block of G_{even} is triangle-free with maximum degree ≤ 3

Maximum degree Δ

- **[Bonamy, Perrett, 2016]**: $\Delta \leq 5$
- **[Chu, Fan, Liu, 2021]**: $\Delta = 6$ when there is no $6-6$ edge

Sparse graphs

- **[Botler, Sambinelli, Coelho, Lee, 2017]**: Treewidth ≤ 3
- **[Botler, Jiménez, Sambinelli, 2018]**: Triangle-free planar graphs

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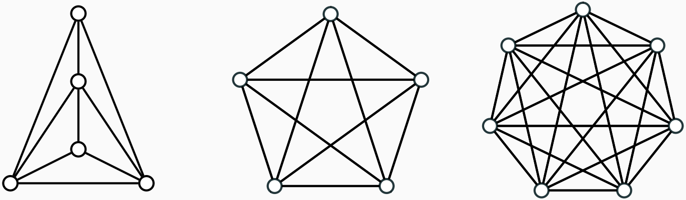
Stronger conjecture

Natural obstructions to the bound $\left\lfloor \frac{n}{2} \right\rfloor$:



K_5^- K_5 K_7 $\cdots$

Odd semi-cliques: cliques on $2k+1$ vertices, delete $\leq k-1$ edges

$=$ graphs with $> \left\lfloor \frac{n}{2} \right\rfloor (n-1)$ edges

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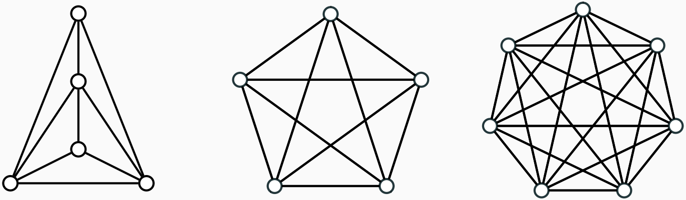

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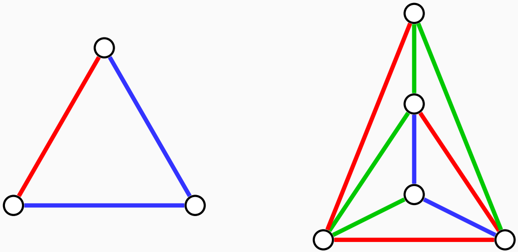
Strong Gallai conjecture [Bonamy, Perrett, 2016]

Every n-vertex connected graph either has a decomposition into $\leq \left\lfloor \frac{n}{2} \right\rfloor$ paths or is an odd semi-clique.

Our contribution

Theorem [B., Bonamy, Bonichon, 2021+]

Every n-vertex connected **planar** graph, different from K_3 and K_5^-, can be decomposed into $\leq \left\lfloor \frac{n}{2} \right\rfloor$ paths.



K_3 K_5^-

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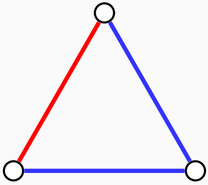
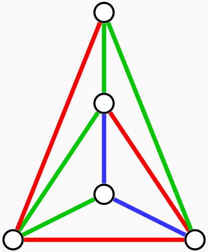


K_3



K_5^-

Corollary

Gallai's conjecture holds on planar graphs.

The proof on planar graphs

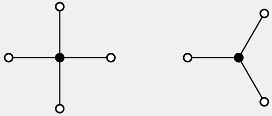
(2021+)

Outline of the proof

Main lemma *(reducibility)*

A **minimum counterexample** to Gallai's conjecture on planar graphs **does not contain** a configuration:

- $\mathcal{C}_I$: 2 vertices of degree ≤ 4



$\mathcal{C}_I$

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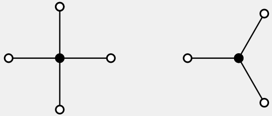

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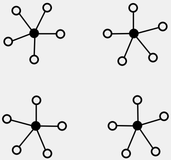

$\mathcal{C}_{II}$

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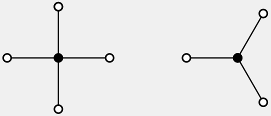

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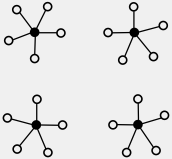

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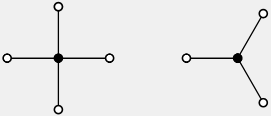

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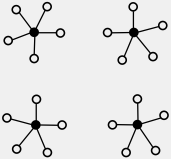

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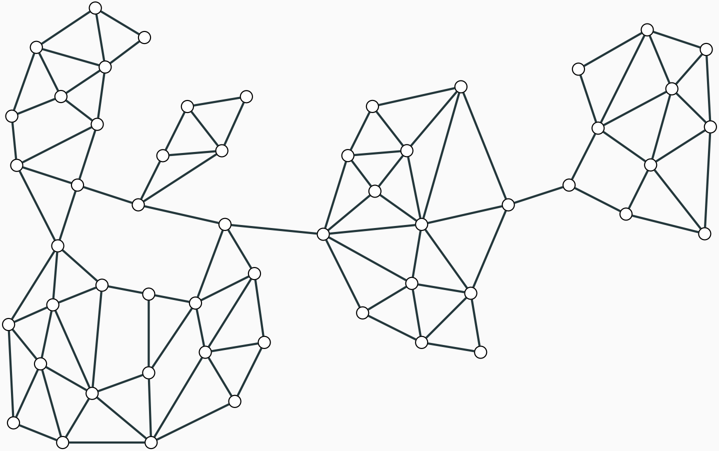
Proof on planar graphs

Part I: $\mathcal{C}_I$ configurations

General idea

$G \equiv$ **minimum counterexample**, n vertices



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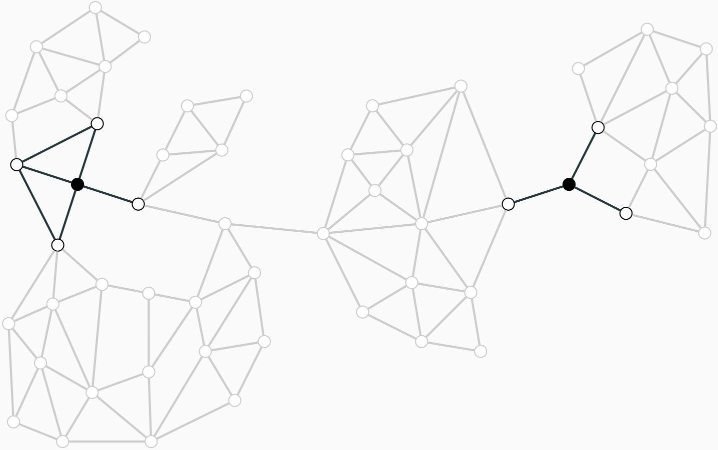

General idea

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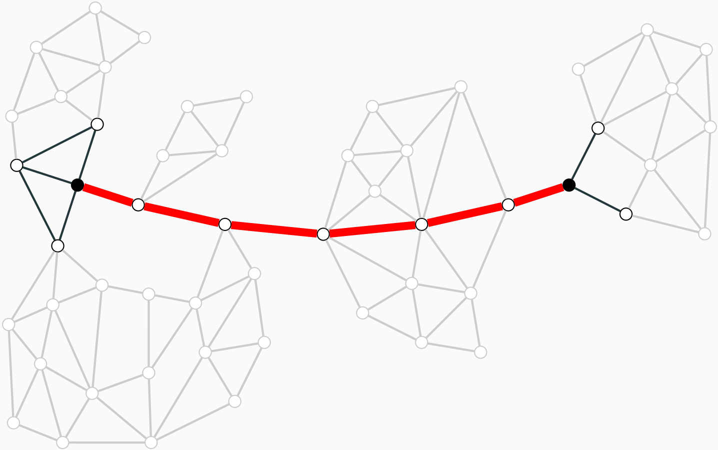

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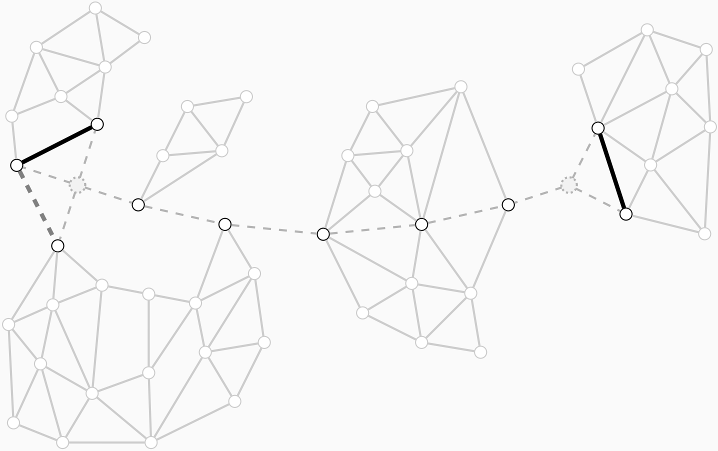

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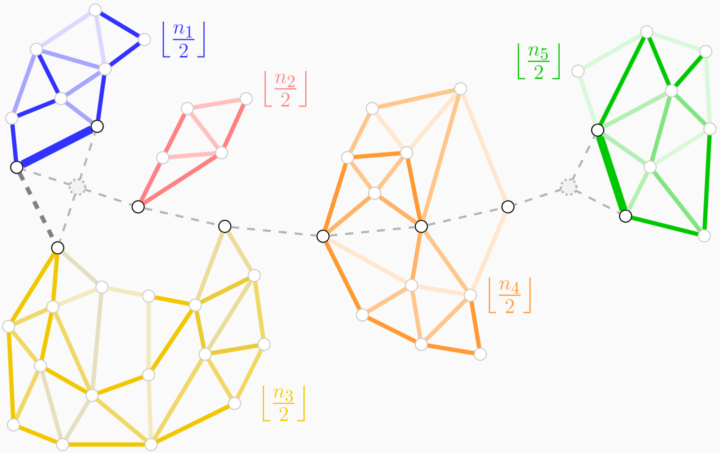

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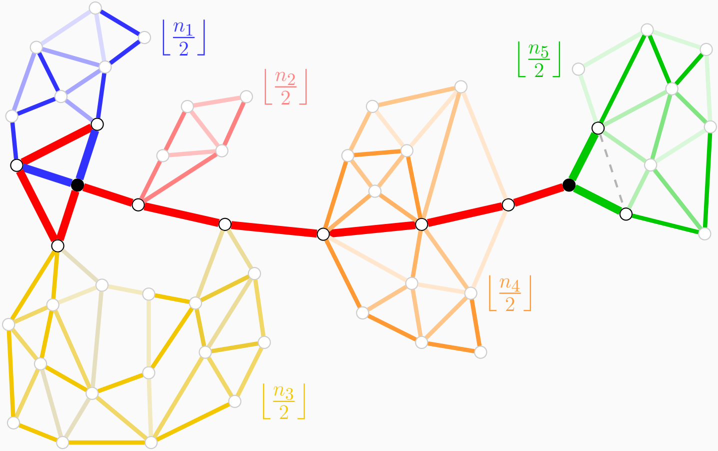

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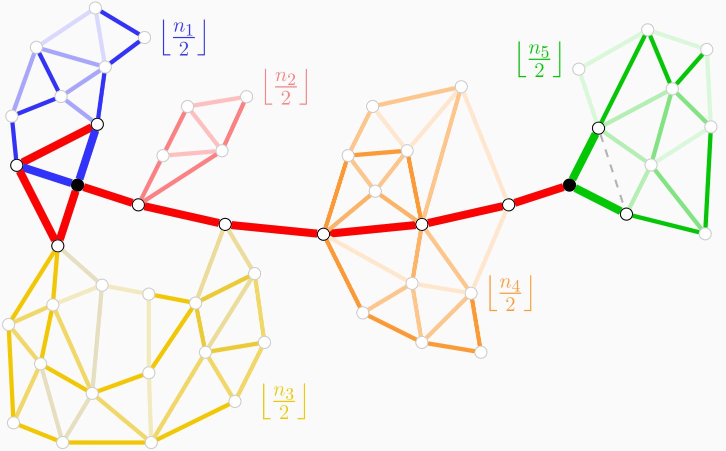
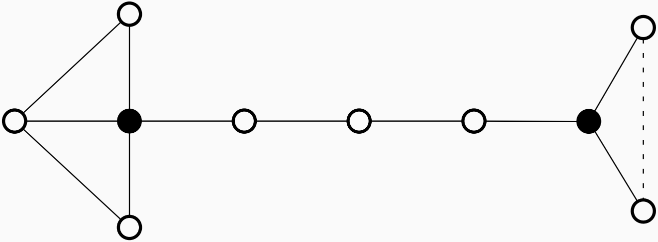


Colors used: $1 + \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \left\lfloor \frac{n_3}{2} \right\rfloor + \left\lfloor \frac{n_4}{2} \right\rfloor + \left\lfloor \frac{n_5}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor$

C_I reduction

When the two special vertices are at distance ≥ 3:



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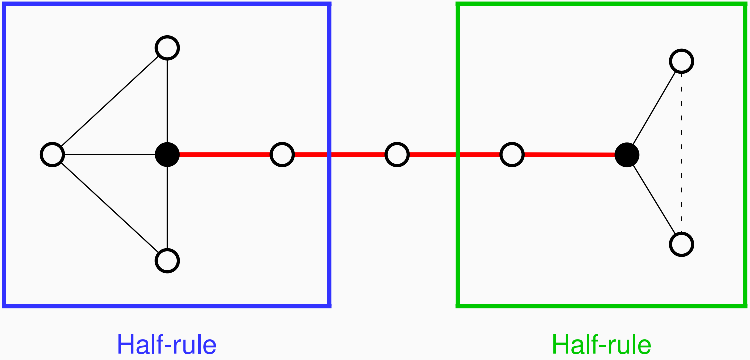

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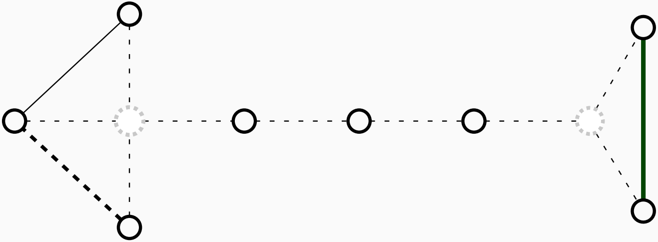

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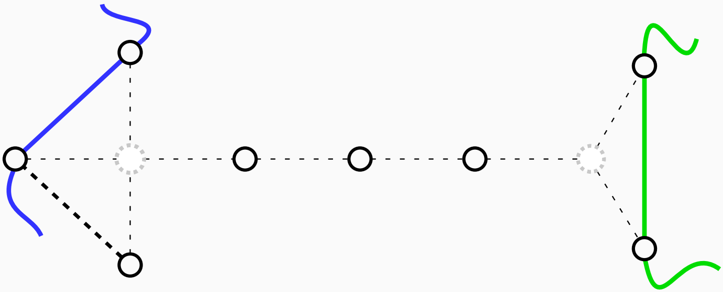

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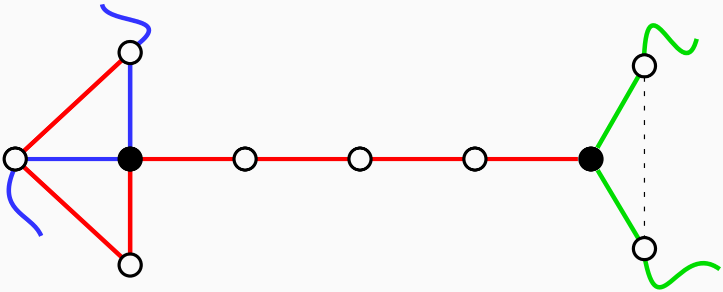
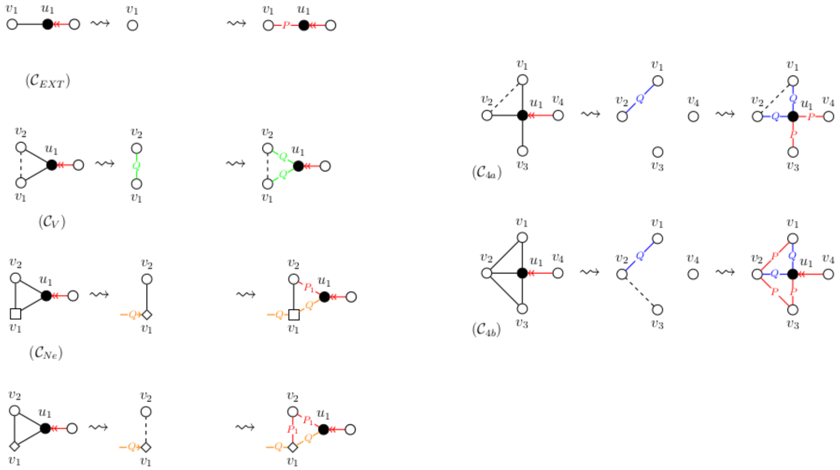
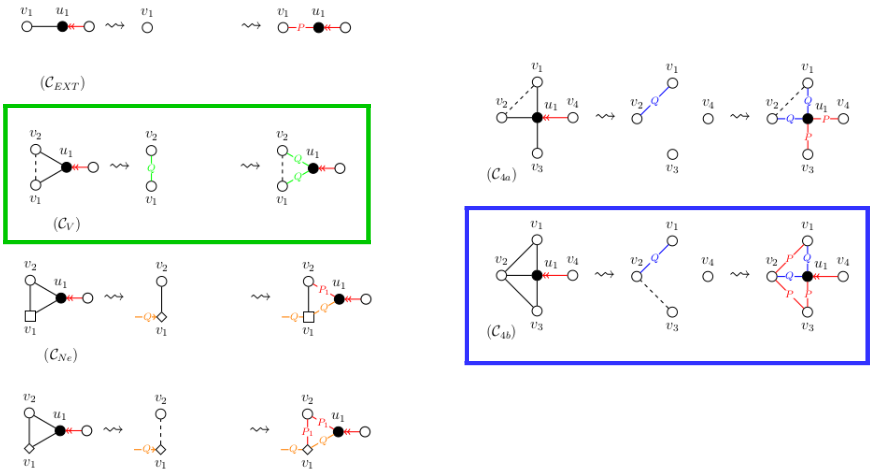


All the half-rules

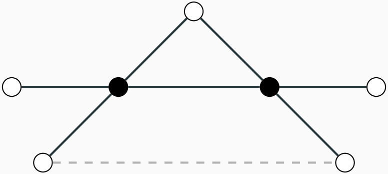


All the half-rules



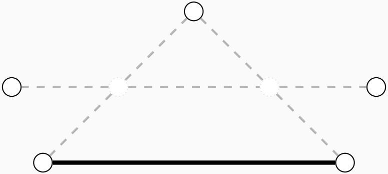
$\mathcal{C}_I$ configurations

When the two special vertices are at distance ≤ 2:



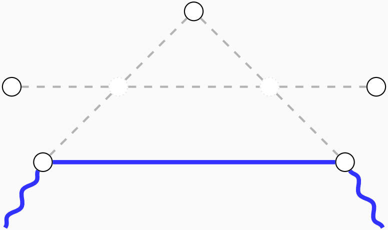
$\mathcal{C}_I$ configurations

When the two special vertices are at distance ≤ 2:



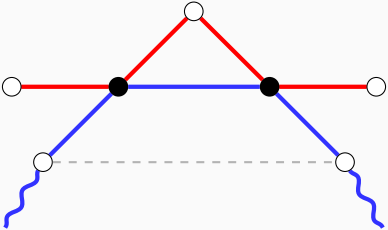
$\mathcal{C}_I$ configurations

When the two special vertices are at distance ≤ 2:



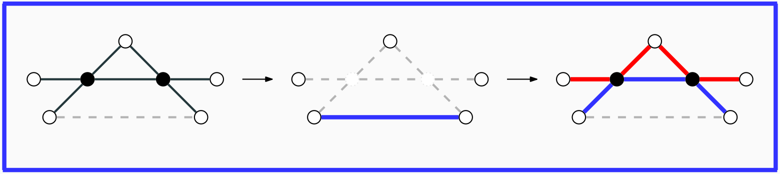
$\mathcal{C}_I$ configurations

When the two special vertices are at distance ≤ 2:

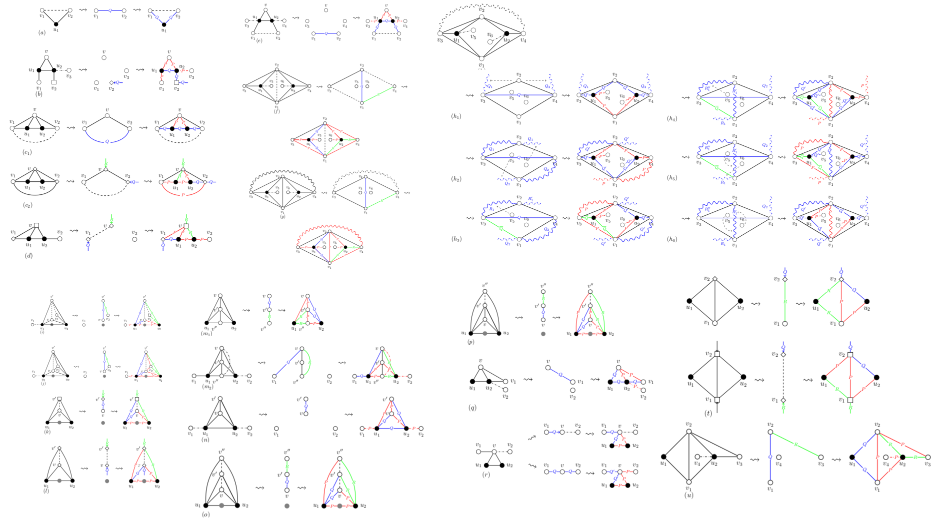


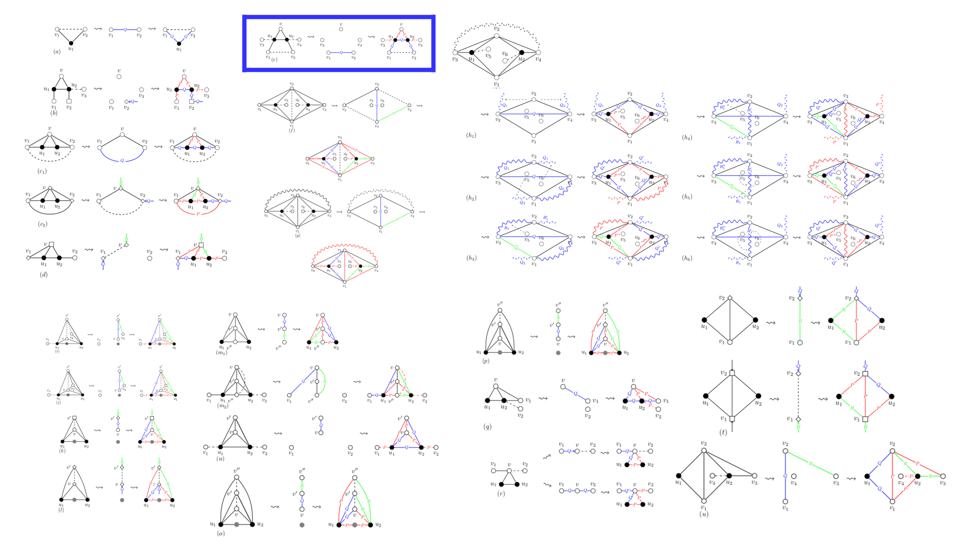
$\mathcal{C}_I$ configurations

When the two special vertices are at distance ≤ 2:



All the full-rules

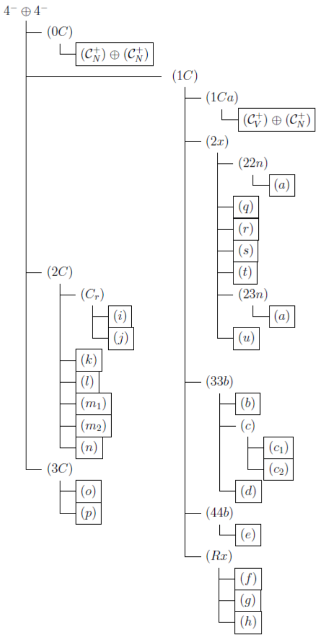




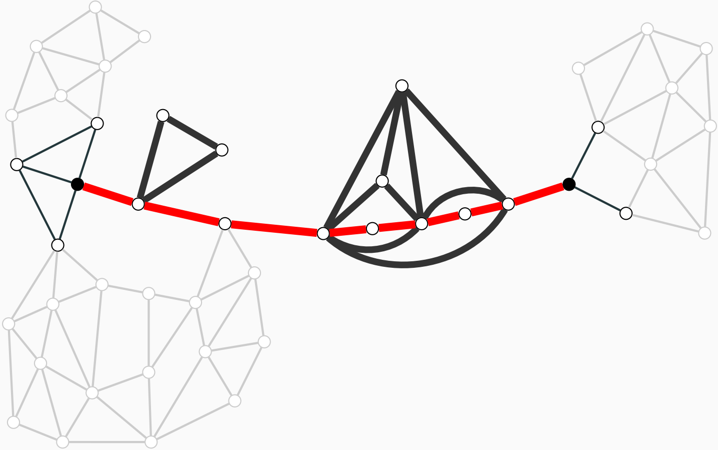
These rules cover all cases

Lemma

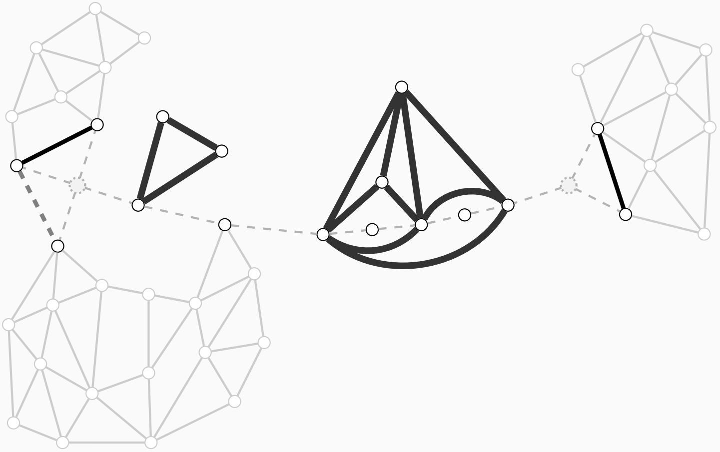
Any C_I configuration can be treated by one of the rules.



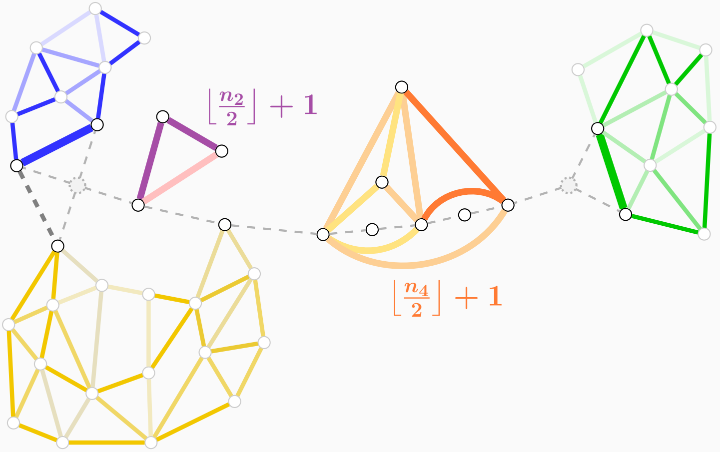
K_3/K_5^- strategy



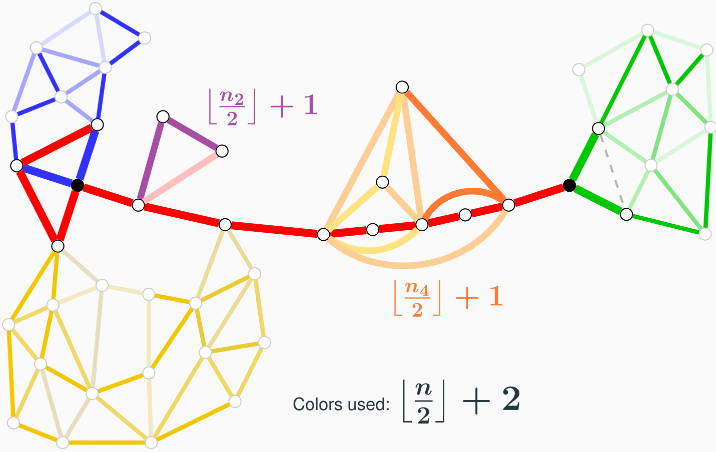
K_3/K_5^- strategy



K_3/K_5^- strategy

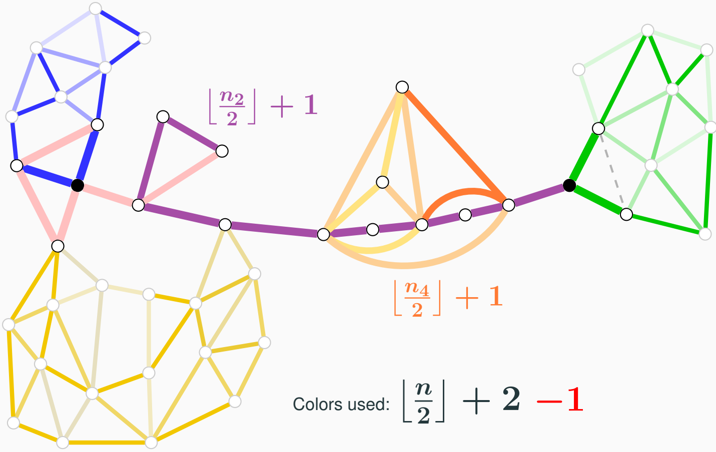


K_3/K_5^- strategy



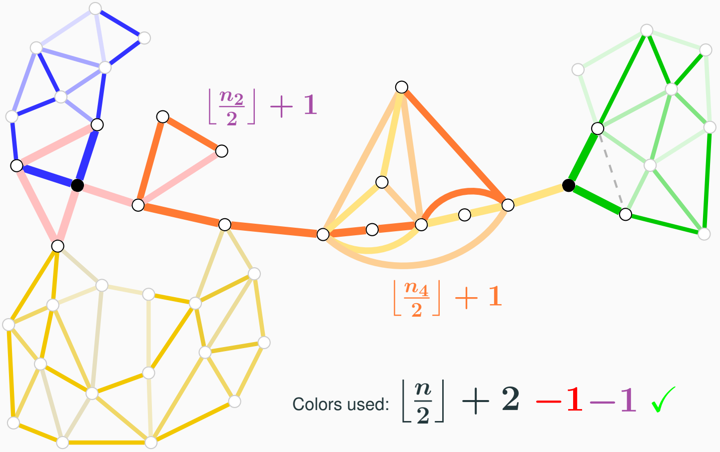
Colors used: $\left\lfloor \frac{n}{2} \right\rfloor + 2$

K_3/K_5^- strategy



Colors used: $\left\lfloor \frac{n}{2} \right\rfloor + 2$ -1

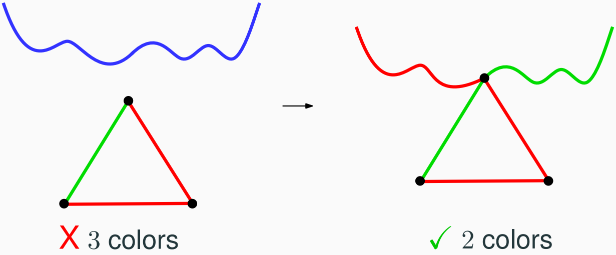
K_3/K_5^- strategy



Colors used: $\left\lfloor \frac{n}{2} \right\rfloor + 2 - 1 - 1$ ✓

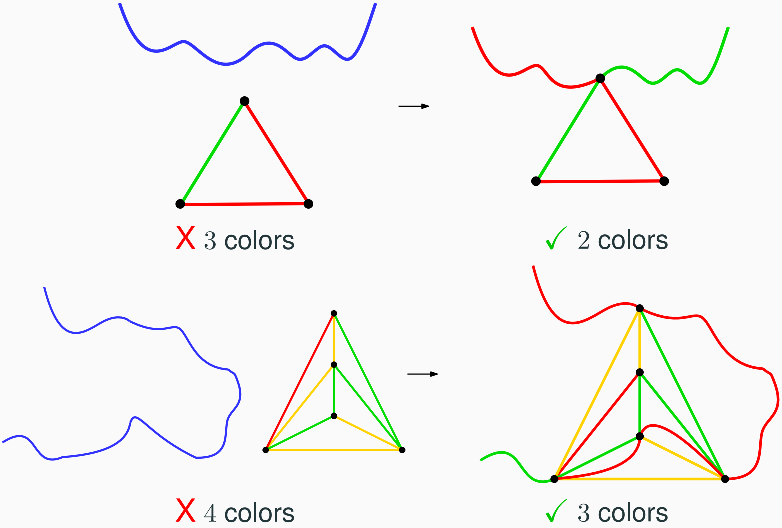
K_3/K_5^- strategy

Combining K_3 and K_5^- components with a path of the decomposition



K_3/K_5^- strategy

Combining K_3 and K_5^- components with a path of the decomposition

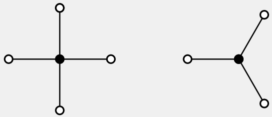


Outline of the proof

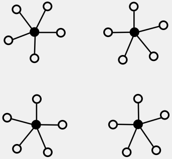
Main lemma *(reducibility)*

A **minimum counterexample** to Gallai's conjecture on planar graphs **does not contain** a configuration:

- $\mathcal{C}_I$: 2 vertices of degree ≤ 4 ✓
- $\mathcal{C}_{II}$: 4 vertices of degree 5 *(with additional connectivity requirements)*



$\mathcal{C}_I$



$\mathcal{C}_{II}$

Final lemma *(unavoidability)*

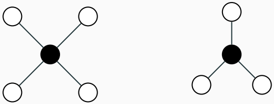
All planar graphs on $n \geq 2$ vertices **contain** a configuration $\mathcal{C}_I$ or $\mathcal{C}_{II}$.

Proof on planar graphs

Part II: $\mathcal{C}_{II}$ configurations

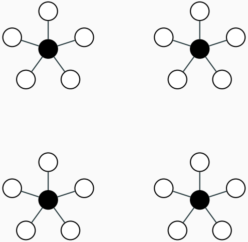
Adapting the method to C_{II} configurations

$\mathcal{C}_I$ configurations



2 special vertices of degree ≤ 4

$\mathcal{C}_{II}$ configurations



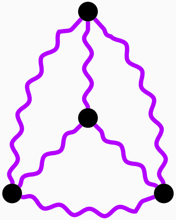
4 special vertices of degree 5

Adapting the method to C_{II} configurations

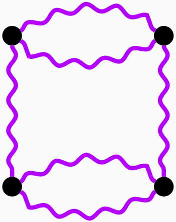
$\mathcal{C}_I$ configurations

A (shortest) path

$\mathcal{C}_{II}$ configurations



K_4-subdivision

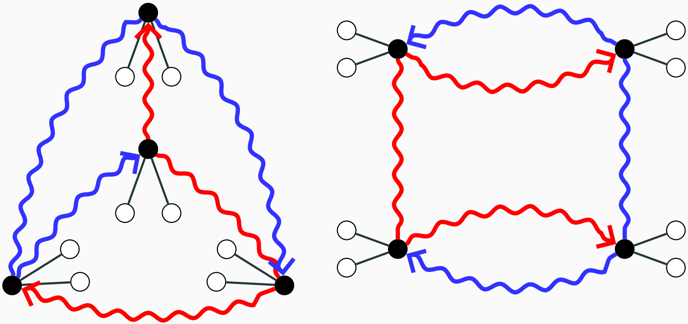


C_{4+}-subdivision

Subdivisions

Theorem [Yu, 1998]

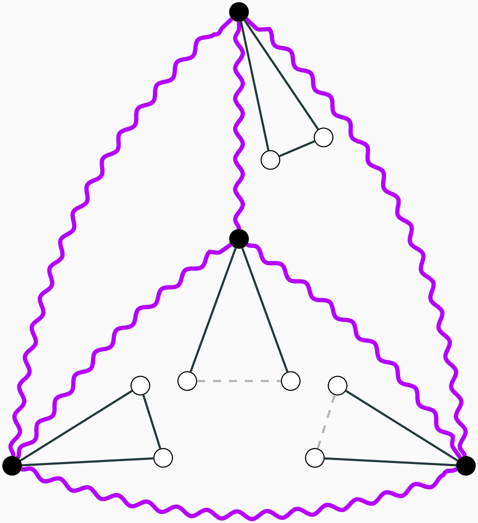
*Under certain connectivity conditions**, a planar graph contains a K_4-subdivision or a C_{4+}-subdivision rooted on 4 given vertices.



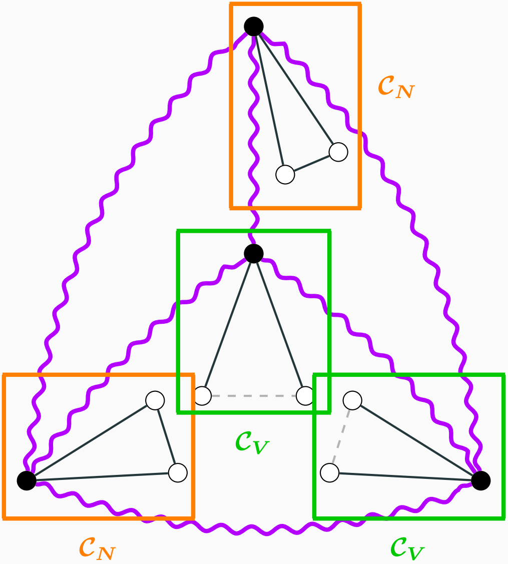
- Decomposable into 2 paths
- One end of path on each special vertex

* No 3-cut separates two special vertices

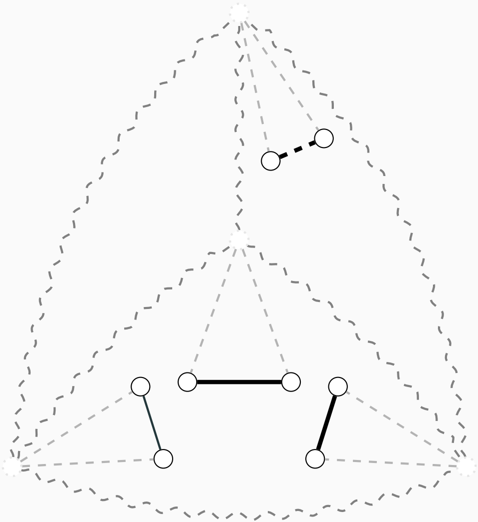
C_{II} reduction



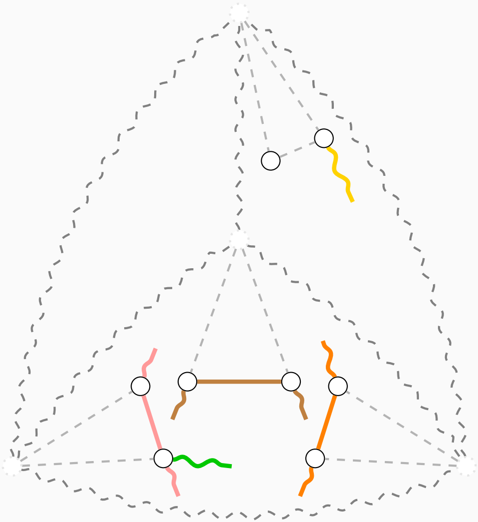
C_{II} reduction



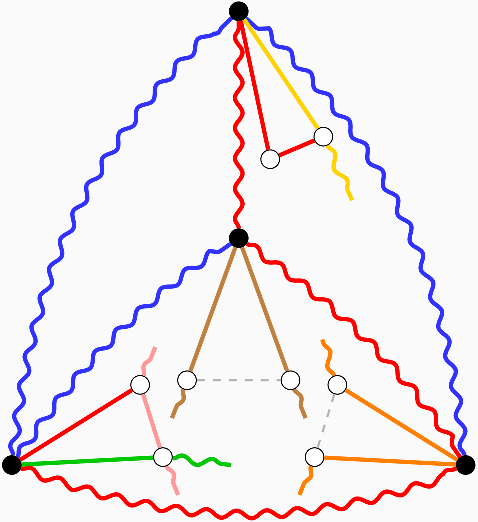
C_{II} reduction



C_{II} reduction

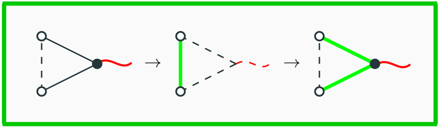


C_{II} reduction

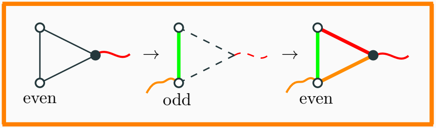


Patterns

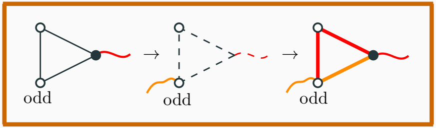
Examples of **patterns**



$\mathcal{C}_V$

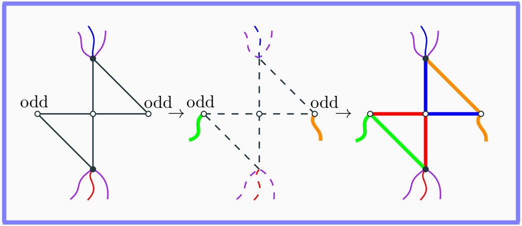


$\mathcal{C}_{Ne}$

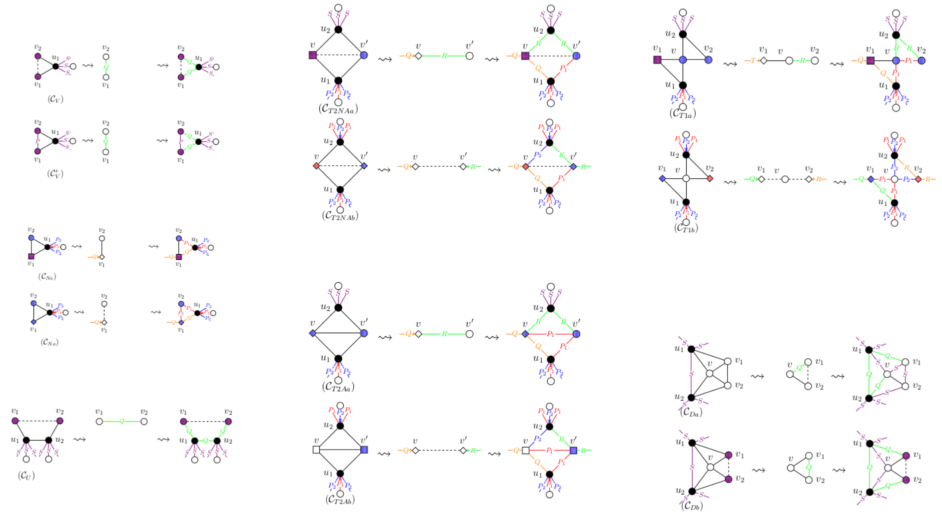


$\mathcal{C}_{No}$

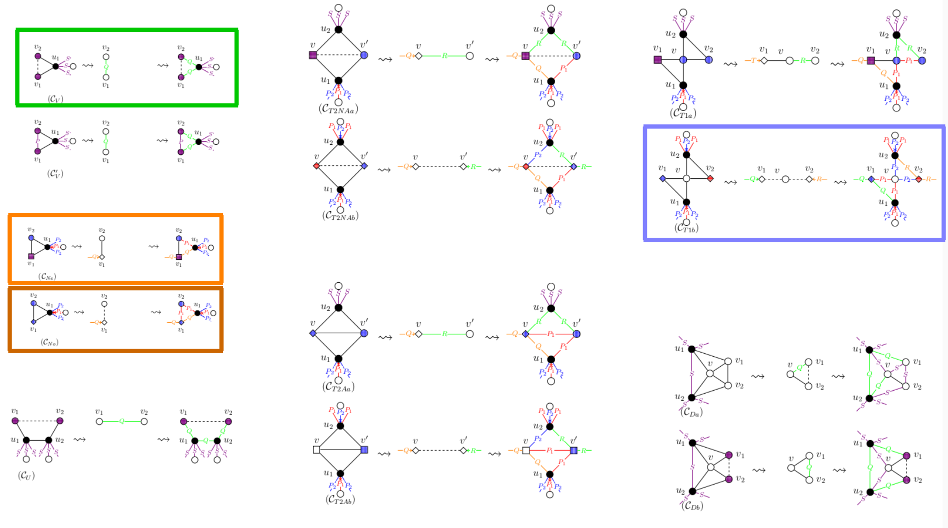
$\mathcal{C}_{T1b}$



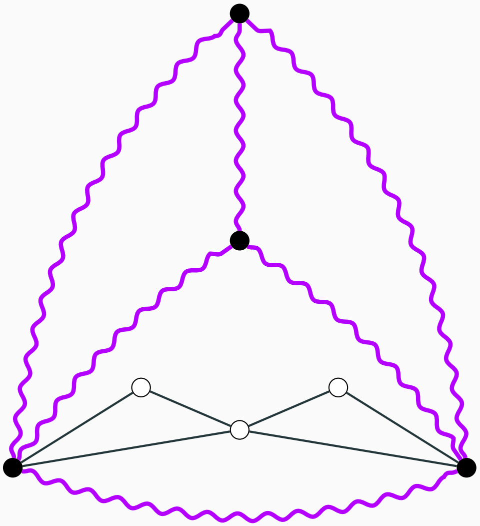
All the patterns



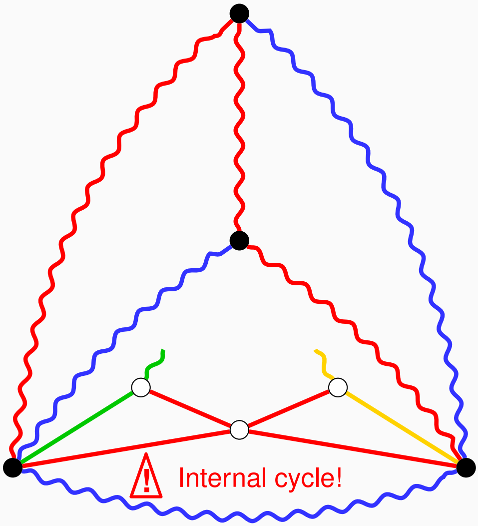
All the patterns



Things that can go wrong

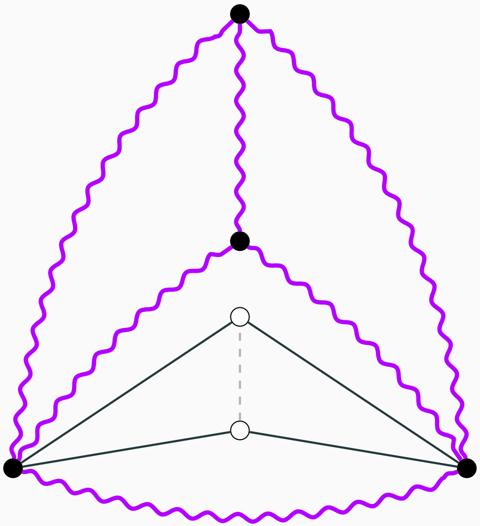


Things that can go wrong



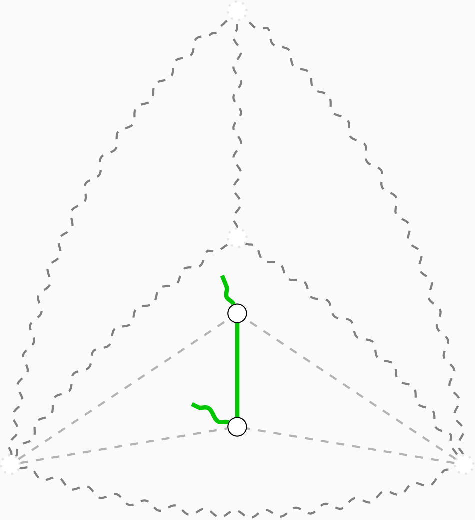
"Close problem"

Things that can go wrong



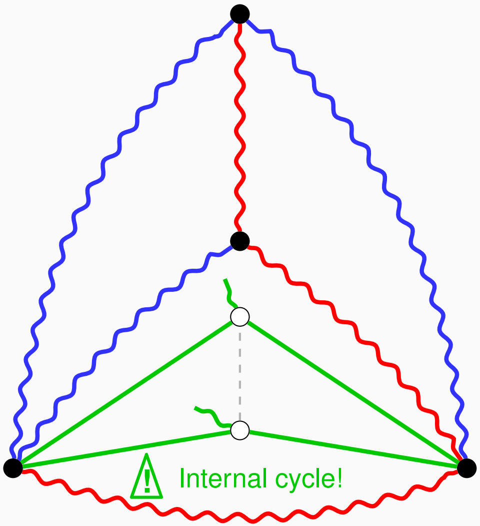
"Close problem"

Things that can go wrong



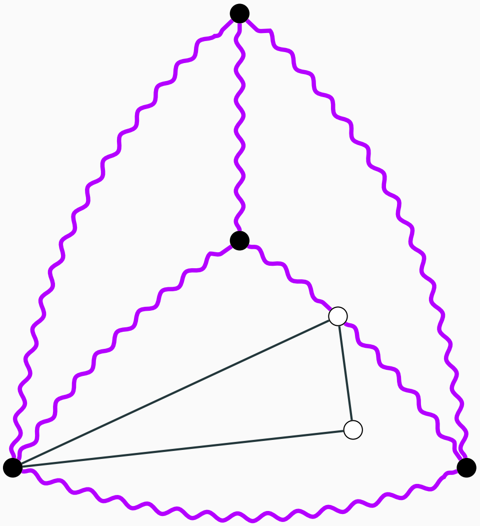
"Close problem"

Things that can go wrong

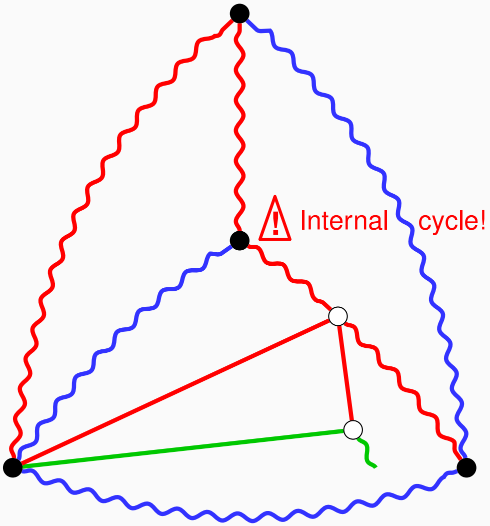


"Close problem"

Things that can go wrong



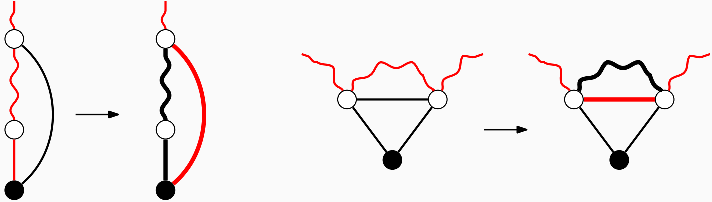
Things that can go wrong



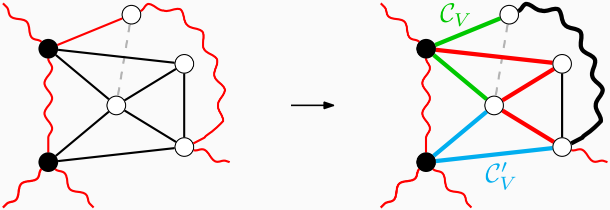
“Distant problem”

Pre-processing

Step 1: Eliminating chords in the subdivision



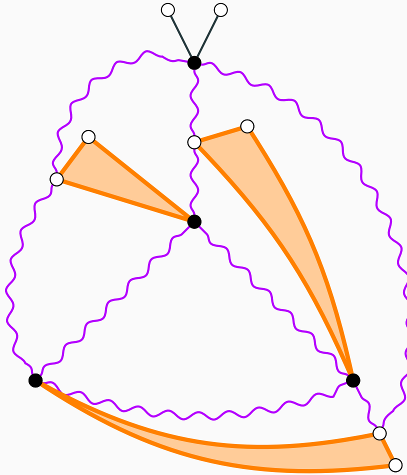
Step 2: Eliminating some configurations by redirection



(4 similar redirection rules)

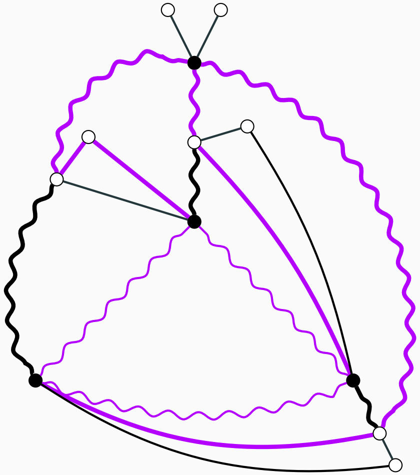
Distant problems

Eliminating **distant problems**:



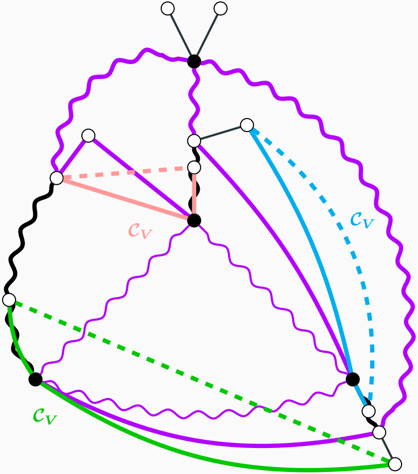
Distant problems

Eliminating **distant problems**:



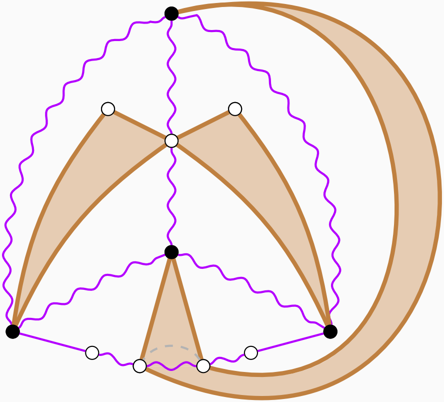
Distant problems

Eliminating **distant problems**:



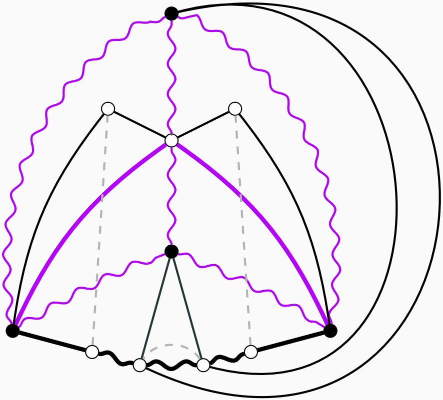
Close problems

Eliminating **close problems**:



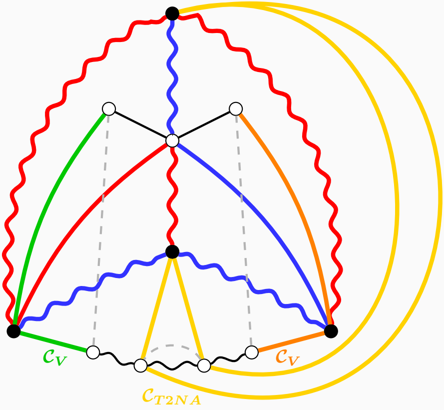
Close problems

Eliminating **close problems**:



Close problems

Eliminating **close problems**:



All the distant and close configurations

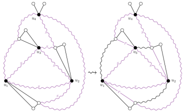


Figure 4.11: Semi-subdivision of D_1

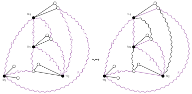


Figure 4.12: Semi-subdivision of D_2

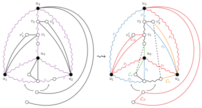


Figure 4.13: Reduction of configuration D_3

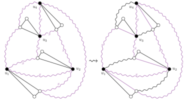


Figure 4.14: Semi-subdivision of D_4

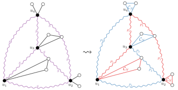


Figure 4.15: J_1 in a case where u_1 and u_3 cause distant problems

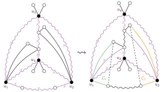


Figure 4.16: J_2 in a case where the length of $u_1 \sim u_2$ is at least 2

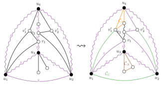


Figure 4.17: Reduction of configuration J_3

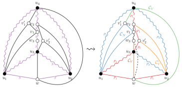


Figure 4.18: Reduction of configuration J_4. Example of a 2-coloring of S'

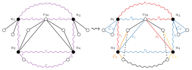


Figure 4.31: Reduction of configuration R_9

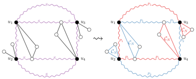


Figure 4.19: J_5 when u_1 and u_4 cause distant problems

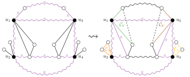


Figure 4.20: Reduction of configuration J_6

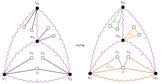


Figure 4.23: Reduction of configuration R_1. u_3, u_4 may cause distant problems

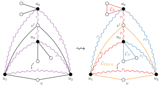


Figure 4.24: Reduction of configuration R_2

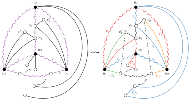


Figure 4.25: R_3 when $v_1 \neq v_2$, $v'_1 \neq v'_2$

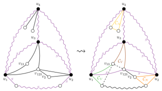


Figure 4.26: R_4 when $l(u_1 \sim u_2) \geq 2$ in S

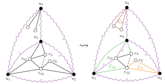


Figure 4.27: Reduction of configuration R_5. The special vertex u_4 may cause a distant problem on $u_1 \sim u_2$ or $u_2 \sim u_3$

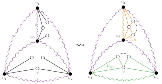


Figure 4.28: R_6 when u_1, u_2 and u_3, u_4 form $\mathcal{C}_{D1}$ configurations

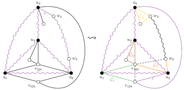


Figure 4.29: R_7 when $l(u_2 \sim u_4) \geq 2$ in S

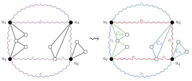
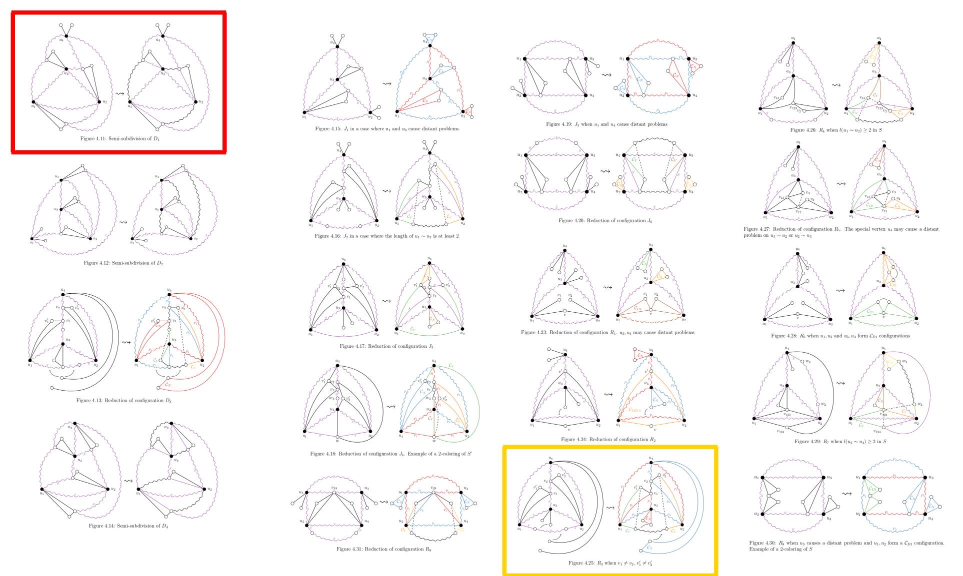


Figure 4.30: R_8 when u_3 causes a distant problem and u_1, u_2 form a $\mathcal{C}_{D1}$ configuration. Example of a 2-coloring of S

All the distant and close configurations



These configurations cover all cases

K_4

- 2 vertices involved in a close problem: u_1, u_2 share a r.n. v
  - $v \notin S$
    - u_1, u_2 share only one r.n.: $\boxed{R_1}$
    - u_1, u_2 share a r.n. v'
      - $v' \notin S$: impossible
      - $v' \in S$: $\boxed{R_2}$
  - $v \in S$
    - u_1, u_2 share a r.n. v'; $v' \notin S$: see case $v \notin S$
    - u_1, u_2 share a r.n. v'; $v' \in S$: $\boxed{R_3}$
    - u_1 r.n. v_1; u_2 r.n. v_2; $v'_1 \neq v'_2$: $\boxed{R_3, J_4}$
- 3 vertices involved in a close problem: u_1, u_2, u_3; u_1, u_3 share a r.n. v_{13}
  - $v_{13} \notin S$
    - $v_1 = v_3$: $\boxed{R_4}$
    - $v_1 \neq v_3$: $\boxed{R_5}$
  - $v_{13} \in S$: Impossible
- 4 vertices involved in close problems
  - Two independent close problems each involving two vertices: u_1, u_2 share one r.n. v_{12}; u_3, u_4 share one r.n. v_{34}
    - $v_{12}, v_{34} \notin S$: $\boxed{R_6}$
    - $v_{12}, v_{34} \in S$: $\boxed{R_3}$
  - One close problem involving all four vertices
    - P_2: u_1, u_2 share two r.n. v_{12}, v'_{12}: $\boxed{R_7}$
    - P_1: u_1, u_2 share one r.n. v_{12}: $\boxed{R_3}$

C_{4+}

- 2 distant problems: impossible
- ≤ 1 distant problem
  - r.n. of 2-linked disjoint: $\boxed{R_8}$
  - Two 2-linked share a r.n.: $\boxed{R_9}$

K_4

- 1 distant problem: $\boxed{J_1}$
- 2 distant problems on different paths: $\boxed{J_1}$
- 2 distant problems on the same path:
  - General case: $\boxed{J_2}$
  - Specific case #1 (contact $\mathcal{C}_U + \mathcal{C}_V$): $\boxed{J_3}$
  - Specific case #2 (contact $\mathcal{C}_{T2NA} + \mathcal{C}_{T2NA}$): $\boxed{J_4}$

C_{4+}

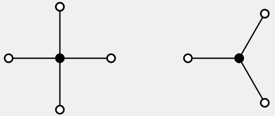
- ≤ 1 distant problem: $\boxed{J_5}$
- 2 problems
  - On different paths: $\boxed{J_5}$
  - On the same path: $\boxed{J_6}$

Outline of the proof

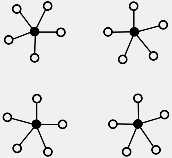
Main lemma *(reducibility)* ✓

A **minimum counterexample** to Gallai's conjecture on planar graphs **does not contain** a configuration:

- $\mathcal{C}_I$: 2 vertices of degree ≤ 4 ✓
- $\mathcal{C}_{II}$: 4 vertices of degree 5 *(with additional connectivity requirements)* ✓



$\mathcal{C}_I$



$\mathcal{C}_{II}$

Final lemma *(unavoidability)*

All planar graphs on $n \geq 2$ vertices **contain** a configuration $\mathcal{C}_I$ or $\mathcal{C}_{II}$.

Proof on planar graphs

Part III: There is no minimum counterexample

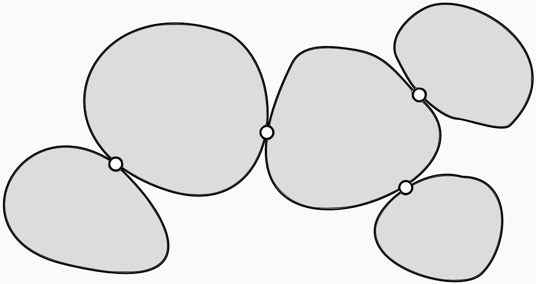
Final proof

Lemma *(unavoidability)*

Any planar graph on $n \geq 2$ vertices **contains** a configuration $\mathcal{C}_I$ or $\mathcal{C}_{II}$.

Goal: We want to find

- a C_I **configuration** (2 vertices of degree 4), OR
- 4 vertices of degree 5 **in a 4-connected component** connected to the rest of the graph with *as few vertices as possible*



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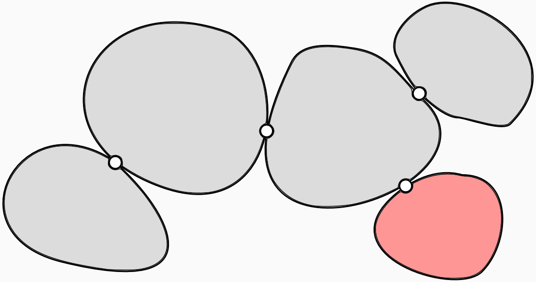

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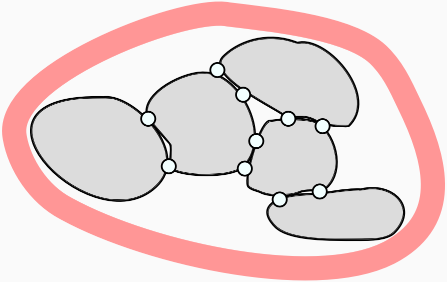

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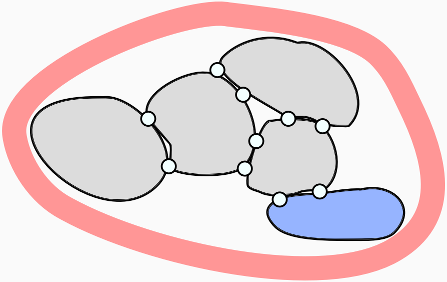

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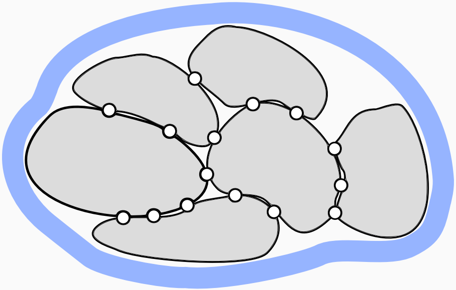

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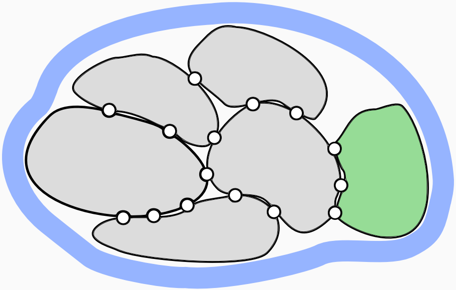
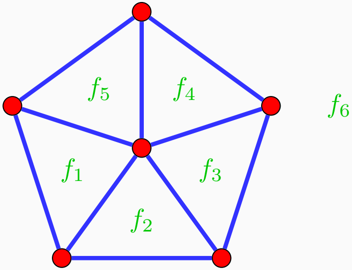


4-connected component

Finding a $\mathcal{C}_{II}$ configuration

Euler's formula (1794)

A connected **planar** graph with vertex set V, edge set E and face set F satisfies: $|V| - |E| + |F| = 2$



$|V| = 6, \ |E| = 10, \ |F| = 6, \ \ |V| - |E| + |F| = 2$

Finding a $\mathcal{C}_{II}$ configuration

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Corollary

A connected planar graph with vertex set V, edge set E and face set F satisfies:

$$2 \cdot \sum_{f \in F} [d(f) - 3] + \sum_{v \in V} [d(v) - 6] = -12$$

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$+$ green component connected by ≤ 3 vertices

$\Downarrow$

If there is **no** $\mathcal{C}_I$ configuration,

there is a $\mathcal{C}_{II}$ configuration with the right connectivity requirements.

Outline of the proof

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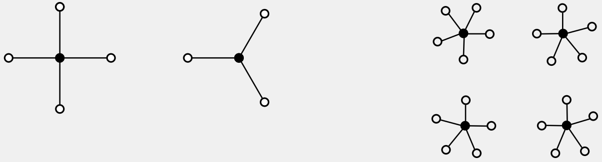

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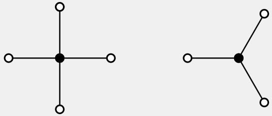
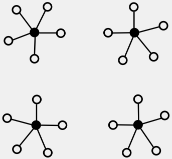


$\mathcal{C}_I$



$\mathcal{C}_{II}$

Final lemma *(unavoidability)* ✓

All planar graphs on $n \geq 2$ vertices **contain** a configuration $\mathcal{C}_I$ or $\mathcal{C}_{II}$.

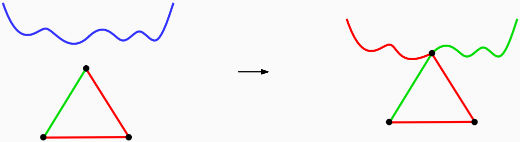
Contradiction $\Rightarrow$ there is no counterexample

Conclusion and further research

Algorithm

- The proof is **constructive**,
  except for Yu's construction of a K_4-subdivision
  - Apply inductively the reduction rules

  - Treat K_3 and K_5^- components by combining them with a path



- Finding a rooted K_4/C_{4+}-subdivision: $O(n^2)$ algorithm
  [Kawarabayashi, Kobayashi, Reed, 2012]

Polynomial-time complexity

Possible extensions

How essential is **planarity** to our proof?

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- Proof built around **Euler's formula** $|V| - |E| + |F| = 2$
  $\rightarrow$ can be generalized to higher **genus**

$$|V| - |E| + |F| = 2 - 2\boldsymbol{g}$$

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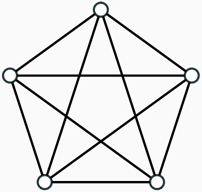
- **Yu's construction** of a K_4-subdivision requires planarity

- Expected growth of the number of cases

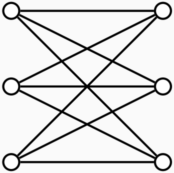
A promising class

Wagner's Theorem [Wagner, 1937]

A graph is **planar** if and only if it has no K_5-minor and no $K_{3,3}$-minor.



K_5

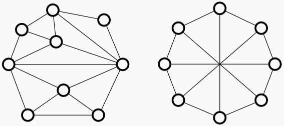


$K_{3,3}$

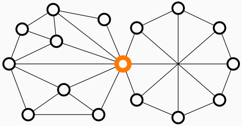
A promising class

Theorem [Wagner, 1937]

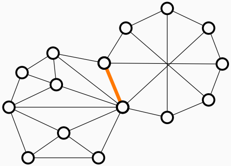
K_5**-minor-free graphs** are the graphs built through 0-, 1- and 2-sums of V_8 and (3-sums of planar graphs)



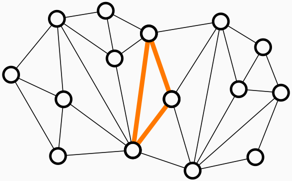
0-sum



1-sum



2-sum



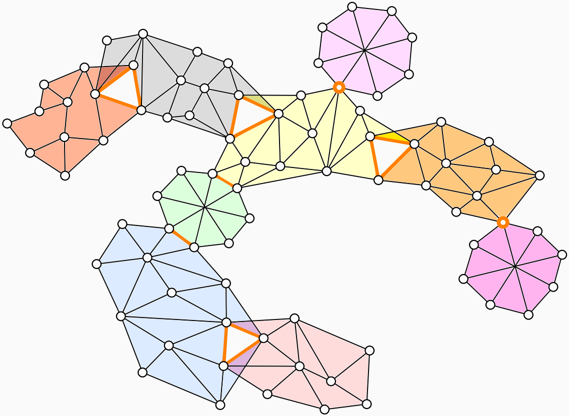
3-sum

A promising class

A K_5-minor-free graph:

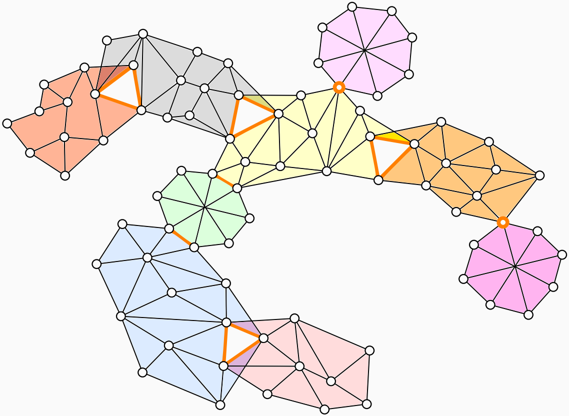
A promising class

A K_5-minor-free graph:



A promising class

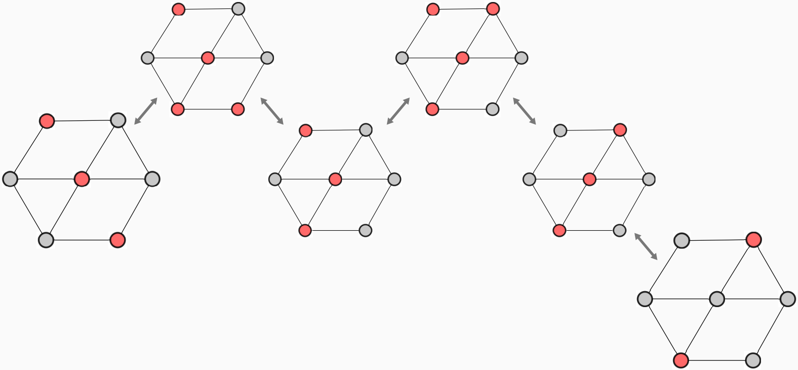
A K_5-minor-free graph:



Thank you for your attention.

Dominating set reconfiguration

Model: **token addition/removal** (TAR)



Optimization problem

OPT-DSR (OPTimization variant of Dominating Set Reconfiguration)

- **Instance** : A graph G, two integers k,s, a dominating set D_0 of size $|D_0| \leq k$.
- **Question** : Is there a dominating set D_s of size $|D_s| \leq s$, such that $D_0 \overset{k}{\leftrightsquigarrow} D_s$?